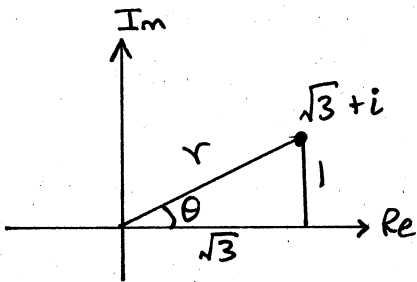


MATH 251 (Winter, 2019)
Term Test 3

by George Ballinger

 Answer the questions in the space provided.
 This test has 5 questions for a total of 25 marks.

1. (4 marks) Evaluate $(\sqrt{3} + i)^{11}$. Express your answer in both polar and rectangular form, using exact values in both cases.



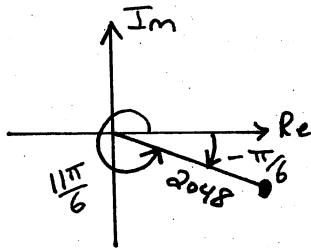
$$r = \sqrt{(\sqrt{3})^2 + 1^2} = 2 \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} \quad \text{or} \quad 30^\circ$$

$$\therefore \sqrt{3} + i = 2e^{i\pi/6}$$

$$(\sqrt{3} + i)^{11} = (2e^{i\pi/6})^{11} = 2^{11} e^{i\frac{11\pi}{6}} = \underline{2048e^{i\frac{11\pi}{6}}}$$

or $2048 \angle \frac{11\pi}{6}$

or $2048 \angle 330^\circ$



(instead of $\frac{11\pi}{6}$ or 330° we could use, for example, $-\frac{\pi}{6}$ or -30°)

In rectangular form

$$2048e^{i\frac{11\pi}{6}} = 2048 \left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6} \right)$$

$$= 2048 \left(\frac{\sqrt{3}}{2} + i \left(\frac{1}{2} \right) \right)$$

$$= \underline{1024\sqrt{3} - 1024i}$$

2. (4 marks) Use Cramer's Rule to find the y -value in the solution of the following system of linear equations. You do not need to solve for the other variables.

$$\begin{cases} 2x - 6z = 10 \\ x + 4y - 2z = -8 \\ 5x + 2y + z = 3 \end{cases}$$

$$\underbrace{\begin{bmatrix} 2 & 0 & -6 \\ 1 & 4 & -2 \\ 5 & 2 & 1 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_b = \underbrace{\begin{bmatrix} 10 \\ -8 \\ 3 \end{bmatrix}}_b$$

$$\det(A) = \begin{vmatrix} 2 & 0 & -6 \\ 1 & 4 & -2 \\ 5 & 2 & 1 \end{vmatrix} = 2 \begin{vmatrix} 4 & -2 \\ 2 & 1 \end{vmatrix} + (-6) \begin{vmatrix} 1 & 4 \\ 5 & 2 \end{vmatrix} = 2(8) - 6(-18) = 124$$

$$\begin{aligned} \det(A_2(\vec{b})) &= \begin{vmatrix} 2 & 10 & -6 \\ 1 & -8 & -2 \\ 5 & 3 & 1 \end{vmatrix} = 2 \begin{vmatrix} -8 & -2 \\ 3 & 1 \end{vmatrix} - 10 \begin{vmatrix} 1 & -2 \\ 5 & 1 \end{vmatrix} + (-6) \begin{vmatrix} 1 & -8 \\ 5 & 3 \end{vmatrix} \\ &= 2(-2) - 10(11) - 6(43) = -372 \end{aligned}$$

$$\therefore y = \frac{\det(A_2(\vec{b}))}{\det(A)} = \frac{-372}{124} = -3$$

3. (5 marks) Let $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$.

(a) Find the eigenvalues and corresponding eigenspaces of A .

(b) If possible, find a diagonal matrix D and an invertible matrix P so that $P^{-1}AP = D$. If it's not possible, then explain why.

$$\begin{aligned} \text{a) } \det(A - \lambda I) &= \begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = (4-\lambda)(2-\lambda) - 3 = 8 - 6\lambda + \lambda^2 - 3 = \lambda^2 - 6\lambda + 5 \\ &= (\lambda-1)(\lambda-5) = 0 \Rightarrow \lambda_1 = 1 \text{ or } \lambda_2 = 5 \quad (\text{eigenvalues}) \end{aligned}$$

$$\lambda_1 = 1 \quad A - \lambda_1 I = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \quad \text{Solve } (A - \lambda_1 I)\vec{x} = \vec{0}$$

$$\left[\begin{array}{cc|c} 3 & 1 & 0 \\ 3 & 1 & 0 \end{array} \right] \rightarrow \begin{array}{l} \frac{1}{3}R_1 \\ R_2 - R_1 \end{array} \left[\begin{array}{cc|c} 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 \end{array} \right] \quad y = t, \quad x = -\frac{1}{3}t$$

$$\therefore \vec{x} = \begin{bmatrix} -\frac{1}{3}t \\ t \end{bmatrix} = \frac{1}{3}t \begin{bmatrix} -1 \\ 3 \end{bmatrix} \quad \therefore \text{an eigenvector is } \vec{x}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$E_{\lambda_1} = \text{span}\left(\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right)$$

$$\lambda_2 = 5 \quad A - \lambda_2 I = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \quad \text{Solve } (A - \lambda_2 I)\vec{x} = \vec{0}$$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 3 & -3 & 0 \end{array} \right] \rightarrow \begin{array}{l} -R_1 \\ R_2 + 3R_1 \end{array} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right] \quad y = t, \quad x = t$$

$$\therefore \vec{x} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \therefore \text{an eigenvector is } \vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$E_{\lambda_2} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

b) Let $D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ and $P = \begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix}$.

4. (5 marks) Consider the following matrix.

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & x \end{bmatrix}$$

For what value(s) of x (if any) is A not diagonalizable? Justify.

Since A is upper triangular, its eigenvalues are the diagonal entries $2, 1,$ and x . If $x \neq 1$ and $x \neq 2$, then A has 3 distinct eigenvalues and \therefore 3 linearly independent eigenvectors and will be diagonalizable.

If $x=1$, then $\lambda=1$ is an eigenvalue of A with algebraic multiplicity 2. A will be diagonalizable iff the geometric multiplicity of $\lambda=1$ is also 2. The same reasoning applies if $x=2$.

Check $x=1$. $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\lambda=1$, $A - \lambda I = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Solve $(A - \lambda I)\vec{x} = \vec{0}$ $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$ (already RREF)

$y=s, z=t, x=-t \quad \therefore \vec{x} = \begin{bmatrix} -t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$E_\lambda = \text{span} \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$. Since geo. mult. of $\lambda=1$ is 2, matrix is diagonalizable

Check $x=2$. $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $\lambda=2$, $A - \lambda I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Solve $(A - \lambda I)\vec{x} = \vec{0}$ $\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \rightarrow -R_2 \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$

$x=t, y=0, z=0 \quad \therefore \vec{x} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$E_\lambda = \text{span} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$. Since geo. mult. of $\lambda=2$ is only 1 then matrix is not diagonalizable

$\therefore \boxed{x=2}$

5. (7 marks) Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 5 \\ -5 \\ 9 \end{bmatrix}$$

and define $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $W = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$.

- Show that B is an orthogonal basis for \mathbb{R}^3 .
- Express \mathbf{v} as a linear combination of the vectors in B . In other words, find $[\mathbf{v}]_B$.
- Find a basis for W^\perp .
- Find $\text{proj}_W(\mathbf{v})$ and $\text{perp}_W(\mathbf{v})$.
- Construct an orthonormal basis for \mathbb{R}^3 using the vectors from B .

Note: If you need more space, you may continue your answer on the back of this page.

$$a) \left. \begin{array}{l} \vec{v}_1 \cdot \vec{v}_2 = 3 - 4 + 1 = 0 \\ \vec{v}_1 \cdot \vec{v}_3 = 2 + 2 - 4 = 0 \\ \vec{v}_2 \cdot \vec{v}_3 = 6 - 2 - 4 = 0 \end{array} \right\} \therefore B \text{ is an orthogonal set. Vectors in } B \text{ are linearly independent, so } B \text{ is an orthogonal basis for } \mathbb{R}^3.$$

$$b) \vec{v} = \underbrace{\left(\frac{\vec{v} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)}_{\text{proj}_{\vec{v}_1}(\vec{v})} \vec{v}_1 + \underbrace{\left(\frac{\vec{v} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\right)}_{\text{proj}_{\vec{v}_2}(\vec{v})} \vec{v}_2 + \underbrace{\left(\frac{\vec{v} \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3}\right)}_{\text{proj}_{\vec{v}_3}(\vec{v})} \vec{v}_3 = \left(\frac{24}{6}\right) \vec{v}_1 + \left(\frac{14}{14}\right) \vec{v}_2 + \left(\frac{-21}{21}\right) \vec{v}_3$$

$$= 4\vec{v}_1 + \vec{v}_2 - \vec{v}_3 \quad \therefore [\vec{v}]_B = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$$

*c) A basis for W^\perp is $\{\vec{v}_3\}$.

$$d) \text{proj}_W(\vec{v}) = \text{proj}_{\vec{v}_1}(\vec{v}) + \text{proj}_{\vec{v}_2}(\vec{v}) = 4\vec{v}_1 + \vec{v}_2 = 4 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \\ 5 \end{bmatrix}$$

$$\text{perp}_W(\vec{v}) = \underbrace{\vec{v} - \text{proj}_W(\vec{v})}_{\text{or proj}_{W^\perp}(\vec{v})} = -\vec{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$

$$e) \|\vec{v}_1\| = \sqrt{6}, \quad \|\vec{v}_2\| = \sqrt{14}, \quad \|\vec{v}_3\| = \sqrt{21}$$

$$\therefore \text{an orthonormal basis for } \mathbb{R}^3 \text{ is } \left\{ \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{14}} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{21}} \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix} \right\}.$$

*Note: In (c) we could find a basis for W^\perp by finding $\text{null}(A^T)$, where $A = [\vec{v}_1 | \vec{v}_2]$. However, since $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal basis for \mathbb{R}^3 and W is spanned by \vec{v}_1 and \vec{v}_2 , then W^\perp will be spanned by the remaining vectors of B , i.e. \vec{v}_3 .