CAMOSUN

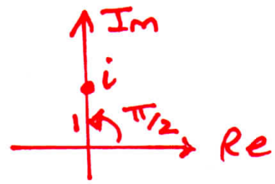
Name: SOLUTIONS

Mark: $\overline{}$ **25**

MATH 251 (Winter, 2026)

Test 3

1. (4 marks) Find all the cube roots of i. Write your answer in both Euler form and rectangular form, using exact values (no decimals).



$$i = e^{i\pi/2}$$

$$z_1 = e^{i\pi/6} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + i\frac{1}{2}$$

$$z_2 = e^{i\left(\frac{\pi}{6}+\frac{2\pi}{3}\right)} = e^{i\frac{5\pi}{6}} = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + i\frac{1}{2}$$

$$z_3 = e^{i\left(\frac{\pi}{6}+\frac{4\pi}{3}\right)} = e^{i\frac{3\pi}{2}} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i$$

2. (7 marks) Let

$$A = \begin{bmatrix} 1 & -5 & 3 & -3 & -3 \\ -4 & -2 & 4 & -2 & -5 \\ -1 & 3 & 5 & -1 & -2 \\ -4 & 4 & 4 & 1 & 5 \\ 3 & -3 & -3 & 0 & 5 \end{bmatrix}, \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 4 & 0 & 0 \\ 0 & 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Verify that $\mathbf{x}_1$ is an eigenvector of A and find its associated eigenvalue, λ_1.

(b) Given that $\lambda_2 = -6$ is an eigenvalue of A, and B is the RREF of $A+6I$, find an eigenvector, $\mathbf{x}_2$, associated with λ_2.

(c) Given that $\lambda_3 = 9$ and $\lambda_4 = 1$ are two additional eigenvalues of A, use the trace of A to find the 5th eigenvalue, λ_5, of A.

(d) Using the eigenvalues of A, compute $\det(A)$.

(e) Is A invertible? Briefly explain.

a) $A\vec{x}_1 = \begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \\ 0 \end{bmatrix} = 4\vec{x}_1 \quad \therefore \lambda_1 = 4$

b) Solve $(A - \lambda_2 I)\vec{x} = \vec{0} \Rightarrow (A+6I)\vec{x} = \vec{0}$. Using RREF B,

$x_3 = t$ (free), $x_1 = -4t$, $x_2 = -5t$, $x_4 = 0$, $x_5 = 0$

$\therefore \vec{x} = t\begin{bmatrix} -4 \\ -5 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \therefore \vec{x}_2 = \begin{bmatrix} -4 \\ -5 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ (letting $t=1$)

c) $tr(A) = 1 + (-2) + 5 + 1 + 5 = 10$

$4 + (-6) + 9 + 1 + \lambda_5 = 10 \Rightarrow \lambda_5 = 2$

d) $\det(A) = 4(-6)(9)(1)(2) = -432$

e) No since $\det(A) \neq 0$

(or since none of the eigenvalues of A are 0)

3. (4 marks) Use Cramer's Rule to solve for x in the following system of equations.

$$\begin{cases} 2x - 3y = 5 \\ x + 6y + 5z = -5 \\ 2x + 3y + z = 8 \end{cases}$$

$$A = \begin{bmatrix} 2 & -3 & 0 \\ 1 & 6 & 5 \\ 2 & 3 & 1 \end{bmatrix} \quad \det(A) = 2\begin{vmatrix} 6 & 5 \\ 3 & 1 \end{vmatrix} - (-3)\begin{vmatrix} 1 & 5 \\ 2 & 1 \end{vmatrix} = 2(-9)+3(-9) = -45$$

$$\vec{b} = \begin{bmatrix} 5 \\ -5 \\ 8 \end{bmatrix} \quad A_1(\vec{b}) = \begin{bmatrix} 5 & -3 & 0 \\ -5 & 6 & 5 \\ 8 & 3 & 1 \end{bmatrix}$$

$$\det(A_1(\vec{b})) = 5\begin{vmatrix} 6 & 5 \\ 3 & 1 \end{vmatrix} - (-3)\begin{vmatrix} -5 & 5 \\ 8 & 1 \end{vmatrix} = 5(-9)+3(-45) = -180$$

$$x = \frac{\det(A_1(\vec{b}))}{\det(A)} = \frac{-180}{-45} = 4$$

4. (5 marks) Suppose A is a 2×2 matrix such that

$$A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3\\3\end{bmatrix} \quad \text{and} \quad A\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}.$$

(a) Find a diagonal matrix D and an invertible matrix P so that $P^{-1}AP = D$.

(b) Find A^5.

a) $A\begin{bmatrix}1\\1\end{bmatrix} = 3\begin{bmatrix}1\\1\end{bmatrix}$ $\lambda_1 = 3,\ \vec{x}_1 = \begin{bmatrix}1\\1\end{bmatrix}$

$A\begin{bmatrix}-1\\1\end{bmatrix} = (-1)\begin{bmatrix}-1\\1\end{bmatrix}$ $\lambda_2 = -1,\ \vec{x}_2 = \begin{bmatrix}-1\\1\end{bmatrix}$

$D = \begin{bmatrix}3 & 0\\0 & -1\end{bmatrix},\ P = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$

b) $A = PDP^{-1} \Rightarrow A^5 = PD^5P^{-1} = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}243 & 0\\0 & -1\end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}$

$= \begin{bmatrix}243 & 1\\243 & -1\end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix}242 & 244\\244 & 242\end{bmatrix} = \begin{bmatrix}121 & 122\\122 & 121\end{bmatrix}$

5. (5 marks) Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ k \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix},$$

where k is a real number, and suppose that $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ forms an orthogonal basis of $\mathbb{R}^3$.

(a) Find k.

(b) Find the coordinate vector $[\mathbf{u}]_{\mathcal{B}}$ of $\mathbf{u}$ with respect to $\mathcal{B}$.

a) $\vec{V}_1 \cdot \vec{V}_2 = 0$ ✓

$\vec{V}_2 \cdot \vec{V}_3 = 0$ ✓

$\vec{V}_1 \cdot \vec{V}_3 = k-2 = 0 \Rightarrow k = 2$

b) $\vec{u} = \left(\frac{\vec{u}\cdot\vec{V}_1}{\vec{V}_1\cdot\vec{V}_1}\right)\vec{V}_1 + \left(\frac{\vec{u}\cdot\vec{V}_2}{\vec{V}_2\cdot\vec{V}_2}\right)\vec{V}_2 + \left(\frac{\vec{u}\cdot\vec{V}_3}{\vec{V}_3\cdot\vec{V}_3}\right)\vec{V}_3 = \frac{10}{3}\vec{V}_1 + \frac{2}{2}\vec{V}_2 + \frac{-4}{6}\vec{V}_3$

$= \frac{10}{3}\vec{V}_1 + \vec{V}_2 - \frac{2}{3}\vec{V}_3 \Rightarrow [\vec{u}]_B = \begin{bmatrix} 10/3 \\ 1 \\ -2/3 \end{bmatrix}$

(Note: Could alternatively RREF $[\vec{V}_1\ \vec{V}_2\ \vec{V}_3 \mid \vec{u}]$, but this doesn't take advantage of orthogonality of B. and it requires a lot more work.)