CAMOSUN

Name: SOLUTIONS

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MATH 251 (Winter, 2026)

Test 2

1. (3 marks) Evaluate $\begin{bmatrix} -5 & 7 \\ 1 & -2 \end{bmatrix}^T - 2\begin{bmatrix} 6 & 9 \\ 7 & -2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix}\begin{bmatrix} 4 & 5 \end{bmatrix} + 4I$.

$$\begin{bmatrix} -5 & 1 \\ 7 & -2 \end{bmatrix} - \begin{bmatrix} 12 & 18 \\ 14 & -4 \end{bmatrix} + \begin{bmatrix} 16 & 20 \\ 20 & 25 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 13 & 31 \end{bmatrix}$$

2. (3 marks) For what value(s) of k (if any) does the product of the matrices $\begin{bmatrix} k & 2 \\ 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 4 & 6 \\ 9 & 1 \end{bmatrix}$ commute?

$$\begin{bmatrix} k & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 4 & 6 \\ 9 & 1 \end{bmatrix} = \begin{bmatrix} 4k+18 & 6k+2 \\ 48 & 22 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 6 \\ 9 & 1 \end{bmatrix}\begin{bmatrix} k & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4k+18 & 32 \\ 9k+3 & 22 \end{bmatrix}$$

$=$ iff $6k+2 = 32 \Rightarrow k = 5$
and
$9k+3 = 48 \Rightarrow k = 5$

$\therefore\ k = 5$

3. (4 marks) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Use the Gauss-Jordan method to find A^{-1}, and then express both A^{-1} and A as products of elementary matrices.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \qquad E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right] \qquad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$$

$$\xrightarrow{R_1 - 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right] \qquad E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = E_3 E_2 E_1 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

and $A = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

(Answers may vary.)

4. (4 marks) Use the LU method to solve the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \underbrace{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 2 & 5 \\ 0 & 3 \end{bmatrix}}_{U} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

$L\vec{y} = \vec{b} \Rightarrow \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \Rightarrow$ $y_1 = 1$

$-4y_1 + y_2 = 5 \Rightarrow -4 + y_2 = 5 \Rightarrow y_2 = 9$

$U\vec{x} = \vec{y} \Rightarrow \begin{bmatrix} 2 & 5 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix} \Rightarrow$ $3x_2 = 9 \rightarrow x_2 = 3$

$2x_1 + 5x_2 = 1 \rightarrow 2x_1 + 15 = 1 \rightarrow x_1 = -7$

$$\therefore \vec{x} = \begin{bmatrix} -7 \\ 3 \end{bmatrix}$$

5. (5 marks) Let $A = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. ← RREF!

(a) Find a basis for row(A).

$\{[1\ 2\ 0\ 0\ -1], [0\ 0\ 1\ 0\ 3]\}$

(b) Find a basis for col(A).

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

(c) Find a basis for null(A).

$x_2 = r,\ x_4 = s,\ x_5 = t$ (free)

$x_1 = -2r + t,\ x_3 = -3t$

$\vec{x} = r\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

$\therefore \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

(d) Find rank(A) and nullity(A).

rank$(A) = 2$ nullity$(A) = 3$

6. (6 marks)

(a) Complete the definition of what it means for $T : \mathbb{R}^n \to \mathbb{R}^m$ to be a **linear** transformation:

$T(\mathbf{u}+\mathbf{v}) =$ $T(\vec{u}) + T(\vec{v})$ for all $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, and

$T(c\mathbf{u}) =$ $cT(\vec{u})$ for all $\mathbf{u}$ in $\mathbb{R}^n$ and for all scalars c.

(b) Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 4x - 5y \\ 3x - 4y \end{bmatrix}.$$

(i) State the standard matrix for T.

$$[T] = \begin{bmatrix} 4 & -5 \\ 3 & -4 \end{bmatrix}$$

(ii) Find the standard matrix for the inverse transformation T^{-1}.

$$[T^{-1}] = \begin{bmatrix} 4 & -5 \\ 3 & -4 \end{bmatrix}^{-1} = \frac{1}{-1}\begin{bmatrix} -4 & 5 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 3 & -4 \end{bmatrix}$$ (same as $[T]$)

(c) Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2y + 3 \\ -x + 2 \end{bmatrix}.$$

(i) Evaluate $T(\mathbf{i})$, $T(\mathbf{j})$ and $T(\mathbf{i}+\mathbf{j})$.

$$T(\hat{\imath}) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$T(\hat{\jmath}) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$T(\hat{\imath}+\hat{\jmath}) = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

(ii) Is T a linear transformation? Briefly explain.

No. $T(\hat{\imath}) + T(\hat{\jmath}) = \begin{bmatrix} 8 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 5 \\ 1 \end{bmatrix} = T(\hat{\imath}+\hat{\jmath})$

Does not satisfy 1st property in part (a).