Linear Transformations

Definition: A transformation (or function or mapping) $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a **linear transformation** if

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, and
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in $\mathbb{R}^n$ and for all scalars c.

Theorem: Let A be an $m \times n$ matrix. Then the transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T(\mathbf{x}) = A\mathbf{x},$$

for all $\mathbf{x}$ in $\mathbb{R}^n$, is a linear transformation.

Proof: Let $\mathbf{u}$ and $\mathbf{v}$ be in $\mathbb{R}^n$ and let c be a scalar. Then

$$\begin{aligned} &T(\mathbf{u} + \mathbf{v}) = A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v}), \text{ and} \\ &T(c\mathbf{u}) = A(c\mathbf{u}) = c(A\mathbf{u}) = cT(\mathbf{u}). \end{aligned}$$

Therefore, T is a linear transformation.

Theorem: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there is an $m \times n$ matrix A, called the **standard matrix of the linear transformation** and often denoted $A = [T]$, such that

$$T(\mathbf{x}) = A\mathbf{x},$$

for all $\mathbf{x}$ in $\mathbb{R}^n$. In particular, if $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ represent the standard basis vectors in $\mathbb{R}^n$, then the columns of A are $T(\mathbf{e}_1), T(\mathbf{e}_2), \ldots, T(\mathbf{e}_n)$.