CAMOSUN

Name: SOLUTIONS

Mark: $\overline{}$ / **25**

MATH 220 (Fall, 2024)

Test 2

1. (3 marks) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for z defined implicitly by $z = 3x^2y + y^2 + z^3 + 2xy$.

$$\underbrace{3x^2y + y^2 + z^3 + 2xy - z}_{F(x,y,z)} = 0$$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = -\frac{6xy+2y}{3z^2-1} = \frac{2y(3x+1)}{1-3z^2}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3x^2+2y+2x}{3z^2-1} = \frac{3x^2+2y+2x}{1-3z^2}$$

2. (3 marks) Find a unit vector $\hat{\mathbf{u}}$ in the direction in which $f(x, y) = y^2e^{2x} + ye^{xy}$ increases most rapidly from the point $(0, 2)$.

$$\vec{\nabla} f(x,y) = \langle 2y^2e^{2x} + y^2e^{xy},\ 2ye^{2x} + xye^{xy} + e^{xy} \rangle$$

$$\vec{\nabla} f(0,2) = \langle 8+4,\ 4+0+1 \rangle = \langle 12, 5 \rangle$$

$$\hat{u} = \frac{\vec{\nabla} f(0,2)}{\|\vec{\nabla} f(0,2)\|} = \frac{\langle 12, 5\rangle}{\sqrt{12^2+5^2}} = \frac{\langle 12, 5\rangle}{13}$$

3. (4 marks) Use Lagrange multipliers to find the minimum value of $f(x, y, z) = 4x^2 + y^2 + z^2$ subject to the constraint $\underbrace{2x - y + z}_{g(x,y,z)} = 4$. At what point(s) is the minimum attained?

$$\begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g \\ g = 4 \end{cases} \Rightarrow \begin{cases} 8x = 2\lambda & \Rightarrow & x = \frac{\lambda}{4} \\ 2y = -\lambda & \Rightarrow & y = -\frac{\lambda}{2} \\ 2z = \lambda & \Rightarrow & z = \frac{\lambda}{2} \\ 2x - y + z = 4 \end{cases}$$

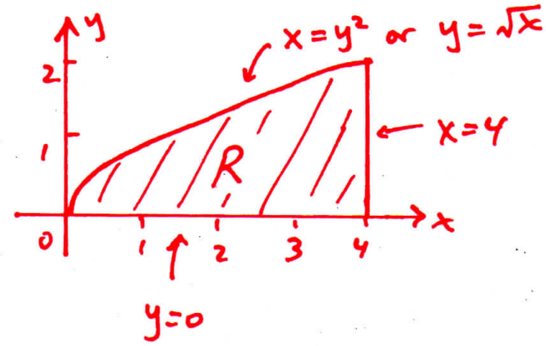
$\therefore\ 2\left(\frac{\lambda}{4}\right) - \left(-\frac{\lambda}{2}\right) + \frac{\lambda}{2} = 4 \Longrightarrow \frac{3\lambda}{2} = 4 \Longrightarrow \lambda = \frac{8}{3}$

Point is $(x, y, z) = \left(\frac{2}{3}, -\frac{4}{3}, \frac{4}{3}\right)$ and min. value is

$f\left(\frac{2}{3}, -\frac{4}{3}, \frac{4}{3}\right) = 4\left(\frac{2}{3}\right)^2 + \left(-\frac{4}{3}\right)^2 + \left(\frac{4}{3}\right)^2 = \frac{16}{9} + \frac{16}{9} + \frac{16}{9} = \frac{16}{3}$

4. (4 marks) Evaluate $\int_0^2 \int_{y^2}^4 4y\sin(x^2)\,dx dy$ by changing the order of integration.

$R: y^2 \le x \le 4,\ 0 \le y \le 2$



$$\int_0^4 \int_0^{\sqrt{x}} 4y \sin(x^2)\, dy\, dx = \int_0^4 2y^2 \sin(x^2) \Big|_0^{\sqrt{x}} dx$$

$$= \int_0^4 2x \sin(x^2)\, dx = -\cos(x^2)\Big|_0^4 = -\cos 16 + 1$$

5. (6 marks) Let Q be the solid region in the first octant that is inside the hemisphere $z = \sqrt{16 - x^2 - y^2}$ and below the cone $z = \sqrt{x^2 + y^2}$. If the solid has variable density $\omega(x, y, z) = k\sqrt{x^2 + y^2 + z^2}$, then find its mass m by setting up and evaluating a triple iterated integral in spherical coordinates.

$\rho = 4$

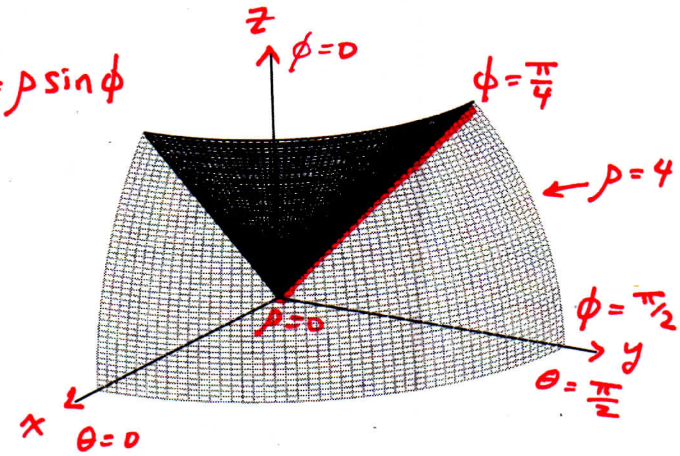
$w = k\rho$

$z = r$

$\rho\cos\phi = \rho\sin\phi$

$\tan\phi = 1$

$\phi = \frac{\pi}{4}$



$$m = \iiint_Q \omega(x,y,z)\, dV = \int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^4 k\rho \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

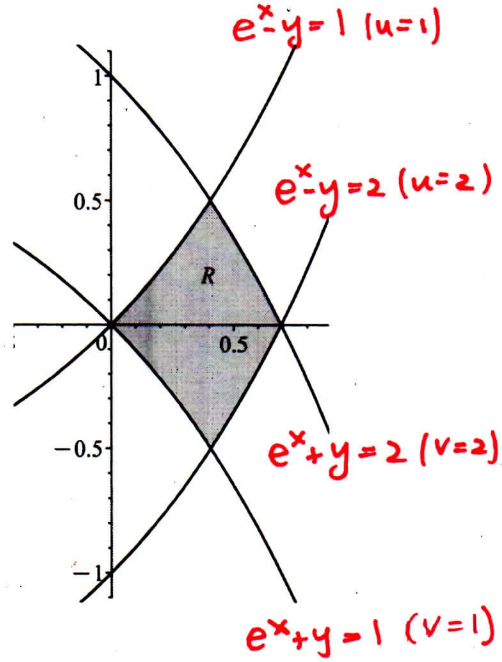
$$= k \int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^4 \rho^3 \sin\phi \, d\rho\, d\phi\, d\theta = k \int_0^{\frac{\pi}{2}} \int_{\pi/4}^{\pi/2} \frac{1}{4}\rho^4 \sin\phi \Big|_0^4 \, d\phi\, d\theta$$

$$= k \int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 64 \sin\phi \, d\phi\, d\theta = 64k \int_0^{\frac{\pi}{2}} -\cos\phi \Big|_{\pi/4}^{\pi/2} d\theta$$

$$= 64k \int_0^{\frac{\pi}{2}} \frac{\sqrt{2}}{2}\, d\theta = 32\sqrt{2}\,k \cdot \theta \Big|_0^{\frac{\pi}{2}} = 32\sqrt{2}\,k \left(\frac{\pi}{2}\right)$$

$$= 16\sqrt{2}\,\pi k$$

6. (5 marks) Use a change of variables to find the volume of the solid region lying below the surface $z = f(x,y) = 6e^x$ and above the region R in the plane bounded by the graphs of $e^x - y = 1$, $e^x - y = 2$, $e^x + y = 1$ and $e^x + y = 2$.



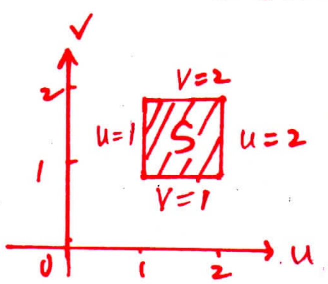
Let $u = e^x - y$ and $v = e^x + y$.

Then $u + v = 2e^x \Rightarrow e^x = \frac{u+v}{2} \rightarrow x = \ln\left(\frac{u+v}{2}\right)$

and $v - u = 2y \rightarrow y = \frac{v-u}{2}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{u+v} & \frac{1}{u+v} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

$$= \frac{1}{2}\left(\frac{1}{u+v}\right) + \frac{1}{2}\left(\frac{1}{u+v}\right) = \frac{1}{u+v}$$



$$V = \iint_R 6e^x \, dx\,dy = \iint_S 6\left(\frac{u+v}{2}\right)\left|\frac{1}{u+v}\right| du\,dv = \int_1^2\int_1^2 3 \, du\,dv = 3$$