CAMOSUN

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MATH 220 (Fall, 2024)

Test 1

1. Let $\mathbf{u} = \langle 2, -3, 6\rangle$ and $\mathbf{v} = \langle 1, 2, -2\rangle$. Find each of the following.

   (a) (1 mark) $\mathbf{u} \times \mathbf{v}$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 6 \\ 1 & 2 & -2 \end{vmatrix} = -6\hat{i} + 10\hat{j} + 7\hat{k} \quad \text{or} \quad \langle -6, 10, 7\rangle$$

   (b) (2 marks) the angle between $\mathbf{u}$ and $\mathbf{v}$, rounded to the nearest degree

$$\cos\theta = \hat{u}\cdot\hat{v} = \frac{\langle 2,-3,6\rangle}{\sqrt{4+9+36}} \cdot \frac{\langle 1,2,-2\rangle}{\sqrt{1+4+4}} = \frac{-16}{\sqrt{49}\,\sqrt{9}} = \frac{-16}{21}$$

$$\therefore \theta = \cos^{-1}\left(\frac{-16}{21}\right) \approx 140^\circ$$

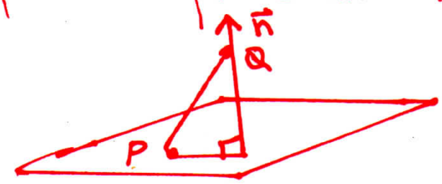
2. (2 marks) Prove that if $\mathbf{u} = \langle u_1, u_2\rangle$ and $\mathbf{v} = \langle v_1, v_2\rangle$ are vectors in $\mathbb{R}^2$ and c is a scalar, then $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$.

$$(c\vec{u})\cdot\vec{v} = (c\langle u_1,u_2\rangle)\cdot\langle v_1,v_2\rangle = \langle cu_1, cu_2\rangle\cdot\langle v_1,v_2\rangle$$

$$= cu_1v_1 + cu_2v_2 = c(u_1v_1+u_2v_2) = c(\langle u_1,u_2\rangle\cdot\langle v_1,v_2\rangle) = c(\vec{u}\cdot\vec{v})$$

3. (3 marks) Find the distance between the point $Q = (1, 2, 3)$ and the plane $2x - 4y + z = 3$.

$P = (0,0,3)$ is point on plane and $\vec{n} = \langle 2, -4, 1 \rangle$ is normal vector



$\vec{PQ} = \langle 1, 2, 0 \rangle$

distance $D = \|\text{proj}_{\vec{n}}(\vec{PQ})\| = |\vec{PQ} \cdot \hat{n}| = \dfrac{|\langle 1,2,0\rangle \cdot \langle 2,-4,1\rangle|}{\sqrt{4+16+1}}$

$= \dfrac{|-6|}{\sqrt{21}} = \dfrac{6}{\sqrt{21}}$ or $\dfrac{2\sqrt{21}}{7}$

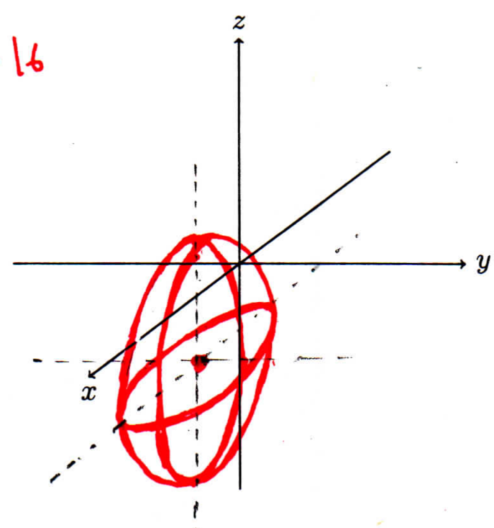
4. (3 marks) Classify and sketch the quadric surface $9x^2 - 18x + 36y^2 + 4z^2 + 16z = 11$. Write the equation in standard form.

$9(x^2 - 2x + 1) + 36y^2 + 4(z^2 + 4z + 4) = 11 + 9 + 16$

$9(x-1)^2 + 36y^2 + 4(z+2)^2 = 36$

$\dfrac{(x-1)^2}{4} + y^2 + \dfrac{(z+2)^2}{9} = 1$

ellipsoid with center $(1, 0, -2)$



5. Convert the equation $r^2 + z^2 = 9$ in cylindrical coordinates to each of the following.

   (a) (1 mark) rectangular coordinates

   $x^2+y^2+z^2=9$

   (b) (1 mark) spherical coordinates.

   $\rho^2=9 \rightarrow \rho=3$

6. (4 marks) Consider the curve represented by $\mathbf{r}(t) = \cos(2t)\mathbf{i} + \cos(2t)\mathbf{j} + \sqrt{2}\sin(2t)\mathbf{k}$. Find $\mathbf{r}(s)$ by reparameterizing the curve with respect to arc length. [Recall $s = \int_0^t \|\mathbf{r}'(u)\|\, du$.]

$\vec{r}\,'(t) = -2\sin(2t)\hat{\imath} - 2\sin(2t)\hat{\jmath} + 2\sqrt{2}\cos(2t)\hat{k}$

$\|\vec{r}\,'(t)\| = \sqrt{4\sin^2(2t)+4\sin^2(2t)+8\cos^2(2t)}$

$= \sqrt{8\sin^2(2t)+8\cos^2(2t)} = \sqrt{8} = 2\sqrt{2}$

$s = \int_0^t 2\sqrt{2}\,du = 2\sqrt{2}u\Big|_0^t = 2\sqrt{2}t \Rightarrow t = \frac{s}{2\sqrt{2}}$

$\therefore \vec{r}(s) = \cos\left(\frac{s}{\sqrt{2}}\right)\hat{\imath} + \cos\left(\frac{s}{\sqrt{2}}\right)\hat{\jmath} + \sqrt{2}\sin\left(\frac{s}{\sqrt{2}}\right)\hat{k}$

7. (4 marks) Find $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ for $\mathbf{r}(t) = 4t\mathbf{i} + 3t\mathbf{j} + t^2\mathbf{k}$.

$\vec{r}'(t) = \langle 4, 3, 2t \rangle$ $\quad \|\vec{r}'(t)\| = \sqrt{16+9+4t^2} = \sqrt{25+4t^2}$

$\vec{a}(t) = \vec{r}''(t) = \langle 0,0,2 \rangle$ $\quad \hat{T}(t) = \dfrac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \dfrac{\langle 4,3,2t \rangle}{\sqrt{25+4t^2}}$

$$a_{\hat{T}} = \vec{a} \cdot \hat{T} = \frac{4t}{\sqrt{25+4t^2}}$$

and $a_{\hat{N}} = \sqrt{\|\vec{a}\|^2 - |a_{\hat{T}}|^2} = \sqrt{4 - \dfrac{16t^2}{25+4t^2}} = \sqrt{\dfrac{100+16t^2-16t^2}{25+4t^2}} = \dfrac{10}{\sqrt{25+4t^2}}$

(OR $\vec{r}' \times \vec{r}'' = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 3 & 2t \\ 0 & 0 & 2 \end{vmatrix} = 6\hat{\imath} - 8\hat{\jmath} = \langle 6, -8, 0 \rangle$

$a_{\hat{N}} = \dfrac{\|\vec{r}' \times r''\|}{\|\vec{r}'\|} = \dfrac{\sqrt{36+64}}{\sqrt{25+4t^2}} = \dfrac{10}{\sqrt{25+4t^2}}$)

Note: $a_{\hat{N}} = \vec{a} \cdot \hat{N}$ is not fun!

8. (2 marks) Let $\mathbf{T}(\phi) = \langle \cos\phi, \sin\phi \rangle$, where $\phi = \phi(s)$. Show that $\kappa = \left|\frac{d\phi}{ds}\right|$. [Recall $\kappa = \left\|\frac{d\mathbf{T}}{ds}\right\|$.]

By chain rule, $\frac{d\hat{T}}{ds} = \frac{d\hat{T}}{d\phi}\frac{d\phi}{ds} = \langle -\sin\phi, \cos\phi \rangle \frac{d\phi}{ds}$

$\therefore \kappa = \left\|\frac{d\hat{T}}{ds}\right\| = \|\langle -\sin\phi, \cos\phi\rangle\| \left|\frac{d\phi}{ds}\right| = \underbrace{\sqrt{\sin^2\phi + \cos^2\phi}}_{1} \left|\frac{d\phi}{ds}\right| = \left|\frac{d\phi}{ds}\right|$

9. (2 marks) Show that the limit $\lim_{(x,y)\to(0,0)} \frac{x^2+2y^2}{x^2+y^2}$ does not exist.

Along $y=0$ limit is $\lim_{x\to 0} \frac{x^2}{x^2} = 1$

Along $x=0$ limit is $\lim_{y\to 0} \frac{2y^2}{y^2} = 2$

different

$\therefore$ limit d.n.e.