Taylor Series

Taylor series for $f(x)$ near $x = x_0$.

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \frac{f^{(4)}(x_0)}{4!}(x - x_0)^4 + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Special Taylor Series.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n \qquad -1 < x < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \qquad -\infty < x < \infty$$

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{(n+1)}(x-1)^n}{n} \qquad 0 < x \le 2$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad -\infty < x < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \qquad -\infty < x < \infty$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} x^{2n+1} \qquad -1 \le x \le 1$$

Taylor series for $f(x, y)$ near $(x, y) = (x_0, y_0)$.

$$\begin{aligned} f(x,y) = {} & f(x_0,y_0) + [f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)] \\ & + \frac{1}{2!}[f_{xx}(x_0,y_0)(x-x_0)^2 + 2f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + f_{yy}(x_0,y_0)(y-y_0)^2] \\ & + \frac{1}{3!}[f_{xxx}(x_0,y_0)(x-x_0)^3 + 3f_{xxy}(x_0,y_0)(x-x_0)^2(y-y_0) \\ & \quad + 3f_{xyy}(x_0,y_0)(x-x_0)(y-y_0)^2 + f_{yyy}(x_0,y_0)(y-y_0)^3] + \cdots \\ & = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{r=0}^{n} \binom{n}{r} \frac{\partial^n f}{\partial x^r \partial y^{n-r}}\Big|_{(x_0,y_0)} (x-x_0)^r (y-y_0)^{n-r}, \end{aligned}$$

where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ are binomial coefficients.