Properties of Vectors

Theorem 1: Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^n$ and let c and d be scalars. Then

(a) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ (addition is commutative)

(b) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ (addition is associative)

(c) $\mathbf{u}+\mathbf{0}=\mathbf{u}$ ($\mathbf{0}$ is additive identity)

(d) $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$ ($-\mathbf{u}$ is additive inverse)

(e) $c(d\mathbf{u})=(cd)\mathbf{u}$ (scalar multiplication is associative)

(f) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$ (distributivity)

(g) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$ (distributivity)

(h) $1(\mathbf{u})=\mathbf{u}$ and $0(\mathbf{u})=\mathbf{0}$

Theorem 2: Let $\mathbf{v}$ be a vector in $\mathbb{R}^n$ and let c be a scalar. Then

(a) $\|\mathbf{v}\|=0$ if and only if $\mathbf{v}=\mathbf{0}$

(b) $\|c\mathbf{v}\|=|c|\|\mathbf{v}\|$

Theorem 3: Let $\mathbf{v}$ be a vector in $\mathbb{R}^n$. If $\mathbf{v}\neq\mathbf{0}$, then $\mathbf{u}=\dfrac{\mathbf{v}}{\|\mathbf{v}\|}=\dfrac{1}{\|\mathbf{v}\|}\mathbf{v}$ is a unit vector in the same direction as $\mathbf{v}$.

Theorem 4: Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^n$ and let c be a scalar. Then

(a) $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$ (commutativity)

(b) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$ (distributivity)

(c) $c(\mathbf{u}\cdot\mathbf{v})=(c\mathbf{u})\cdot\mathbf{v}=\mathbf{u}\cdot c(\mathbf{v})$ (associatvity)

(d) $\mathbf{0}\cdot\mathbf{v}=0$

(e) $\mathbf{v}\cdot\mathbf{v}=\|\mathbf{v}\|^2$ (or equivalently, $\|\mathbf{v}\|=\sqrt{\mathbf{v}\cdot\mathbf{v}}$)

Theorem 5 (Triangle Inequality): For all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$,

$$\|\mathbf{u}+\mathbf{v}\|\leq\|\mathbf{u}\|+\|\mathbf{v}\|$$