CAMOSUN COLLEGE

MATHEMATICS 220

Taylor Series for Functions of Two Variables.

Taylor Series for Functions of Two Variables

There are Taylor series formulae for functions of any number of variables. Here we will derive the formula for Taylor series for functions of two variables.

Recall the Taylor series for a smooth function of one variable $f(x)$ near $x = x_0$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

To find the Taylor series formula for a function of two variables, we can use the Taylor series for a function of one variable and the following trick. Given a smooth function of two variables $f(x, y)$, let

$$F(t) = f(x_0 + t\Delta x, y_0 + t\Delta y)$$

for a fixed Δx and Δy and $0 \le t \le 1$. Then $F(0) = f(x_0, y_0)$ and $F(1) = f(x_0 + \Delta x, y_0 + \Delta y) = f(x, y)$. The Taylor series for the single variable function $F(t)$ near $t = 0$ would be

$$F(t) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} t^n.$$

To find the derivatives of F we apply the chain rule to $f(x_0 + t\Delta x, y_0 + t\Delta y)$:

$$\begin{aligned} \frac{dF}{dt} &= \frac{d}{dt} f(x_0 + t\Delta x, y_0 + t\Delta y) \\ &= \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y \end{aligned}$$

Continuing

$$\begin{aligned} \frac{d^2F}{dt^2} &= \frac{d}{dt}\left(\frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y\right) \\ &= \frac{\partial^2 f}{\partial x^2}\Delta x^2 + 2\frac{\partial^2 f}{\partial x \partial y}\Delta x \Delta y + \frac{\partial^2 f}{\partial y^2}\Delta y^2 \end{aligned}.$$

We see the binomial pattern here and in general, we would have

$$\frac{d^nF}{dt^n} = \sum_{r=0}^{n} \binom{n}{r} \frac{\partial^n f}{\partial x^r \partial y^{n-r}} \Delta x^r \Delta y^{n-r}$$

where all derivatives of f are evaluated at $(x_0 + t\Delta x, y_0 + t\Delta y)$. We can now calculate $F(1)$. Since $t = 0$ in the Taylor series derivatives for $F(t)$, the derivatives of f are now evaluated at (x_0, y_0) and we have

$$\begin{aligned} F(1) &= \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{r=0}^{n} \binom{n}{r} \left.\frac{\partial^n f}{\partial x^r \partial y^{n-r}}\right|_{(x_0, y_0)} \Delta x^r \Delta y^{n-r} \end{aligned}$$

Taylor Series for Functions of Two Variables

Since $F(1) = f(x, y)$ and $\Delta x = x - x_0$, $\Delta y = y - y_0$, we have the Taylor formula for $f(x, y)$ near (x_0, y_0)

$$f(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{r=0}^{n} \binom{n}{r} \left. \frac{\partial^n f}{\partial x^r \partial y^{n-r}} \right|_{(x_0,y_0)} (x - x_0)^r (y - y_0)^{n-r}$$

The first term is just $f(x_0, y_0)$ which is just the height of the surface at (x_0, y_0). Taking the first two terms gives the (linear) approximation

$$f(x,y) \approx f(x_0, y_0) + \left. \frac{\partial f}{\partial x} \right|_{(x_0,y_0)} (x - x_0) + \left. \frac{\partial f}{\partial y} \right|_{(x_0,y_0)} (y - y_0)$$

which is the tangent plane to the surface at (x_0, y_0). Taking the first three terms gives the approximation

$$\begin{aligned} f(x,y) \; &\approx \; f(x_0, y_0) + \left. \frac{\partial f}{\partial x} \right|_{(x_0,y_0)} (x - x_0) + \left. \frac{\partial f}{\partial y} \right|_{(x_0,y_0)} (y - y_0) \\ &+ \frac{1}{2}\left[\left. \frac{\partial^2 f}{\partial x^2} \right|_{(x_0,y_0)} (x - x_0)^2 + 2 \left. \frac{\partial^2 f}{\partial x \partial y} \right|_{(x_0,y_0)} (x_0, y_0)(y - y_0) + \left. \frac{\partial^2 f}{\partial y^2} \right|_{(x_0,y_0)} (y - y_0)^2 \right] \end{aligned}$$

which is the "best fit" quadric surface at (x_0, y_0).

The above trick can also be used to find the Taylor remainder. For the original $F(t)$, the Taylor remainder after n terms would be

$$R_n(t) = \frac{F^{(n+1)}(u)}{(n+1)!} t^n.$$

Recall that $R_n(t)$ gives the value of the remaining terms of the Taylor series for some u with $0 < u < t$. We took $t = 1$ to get the series for $f(x, y)$ and so

$$R_n(x,y) = R_n(1) = \frac{F^{(n+1)}(u)}{(n+1)!}$$

for some $0 < u < 1$.

Since $F(u) = f(x_0 + u\Delta x, y_0 + u\Delta y)$, it follows that the remainder after n terms of the series for $f(x, y)$ would be

$$R_n(x,y) = \frac{1}{(n+1)!} \sum_{r=0}^{n+1} \binom{n+1}{r} \left. \frac{\partial^{n+1} f}{\partial x^r \partial y^{n-r+1}} \right|_{(x_0+u\Delta x, y_0+u\Delta y)} (x - x_0)^r (y - y_0)^{n-r+1}$$

for some $0 < u < 1$. Since $(x_0 + u\Delta x, y_0 + u\Delta y)$ for $0 < u < 1$ is a line segment joining the points (x_0, y_0) and $(x_0 + \Delta x, y_0 + \Delta y)$, there is a point on the line segment where this formula gives the exact remainder.

Taylor Series for Functions of Two Variables

Example 1: Find the Taylor series for $f(x, y) = e^{x+y}$ near $(0, 0)$.

This is too easy since $\dfrac{\partial^n e^{x+y}}{\partial x^r \partial y^{n-r}} = e^{x+y}$ and so all derivatives at $(0, 0)$ are just 1.

$$
\begin{aligned}
e^{x+y} &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{r=0}^{n} \binom{n}{r} \left.\frac{\partial^n f}{\partial x^r \partial y^{n-r}}\right|_{(x_0,y_0)} (x-x_0)^r (y-y_0)^{n-r} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{r=0}^{n} \binom{n}{r} \left.\frac{\partial^n e^{x+y}}{\partial x^r \partial y^{n-r}}\right|_{(0,0)} x^r y^{n-r} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} \\
&= \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!}
\end{aligned}
$$

where the binomial formula $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$ was used to get the last line. The same result could have been reached by multiplying together the two single variable series.

$$
\begin{aligned}
e^{x+y} &= e^x e^y \\
&= \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{y^n}{n!}\right) \\
&= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{x^r}{r!} \frac{y^{n-r}}{(n-r)!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} x^r y^{n-r} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} \\
&= \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!}
\end{aligned}
$$

Where we have used the binomial formula again and the formula for multiplying series

$$
\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} \sum_{r=0}^{n} a_r b_{n-r}
$$

Example 2: Find the Taylor series for $f(x, y) = \dfrac{1}{1-x-y}$ near $(0, 0)$.

Taylor Series for Functions of Two Variables

$$\frac{1}{1-x-y} = \frac{1}{1-(x+y)} = \sum_{n=0}^{\infty}(x+y)^n$$

Notice that the convergence would be for $-1 < x+y < 1$ since we used the geometric series

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n \text{ for } |r| < 1.$$

Example 3: Find the second degree Taylor series approximation to $f(x,y) = \sqrt{x^2+y^3}$ near $(1,2)$ and use it to estimate $\sqrt{(1.02)^2+(1.97)^3}$.

First we need to calculate all of the derivatives up to 2nd order at $(x_0, y_0) = (1,2)$.

$$\begin{array}{lcllcl}
f(x,y) & = & \sqrt{x^2+y^3} & f(1,2) & = & 3 \\
f_x(x,y) & = & \dfrac{x}{\sqrt{x^2+y^3}} & f_x(1,2) & = & \dfrac{1}{3} \\
f_y(x,y) & = & \dfrac{3y^2}{2\sqrt{x^2+y^3}} & f_y(1,2) & = & 2 \\
f_{xx}(x,y) & = & \dfrac{y^3}{(x^2+y^3)^{3/2}} & f_{xx}(1,2) & = & \dfrac{8}{27} \\
f_{xy}(x,y) & = & \dfrac{-3xy^2}{2(x^2+y^3)^{3/2}} & f_{xy}(1,2) & = & -\dfrac{2}{9} \\
f_{yy}(x,y) & = & \dfrac{12x^2y+3y^4}{4(x^2+y^3)^{3/2}} & f_{yy}(1,2) & = & \dfrac{2}{3}
\end{array}$$

and so

$$\begin{aligned}
f(x,y) &\approx f(x_0,y_0)+f_x(x_0,y_0)\Delta x+f_y(x_0,y_0)\Delta y + \frac{1}{2}\left[f_{xx}(x_0,y_0)\Delta x^2+2f_{xy}(x_0,y_0)\Delta x\Delta y+f_{yy}(x_0,y_0)\Delta y^2\right] \\
&\approx 3+\frac{1}{3}(x-1)+2(y-2)+\frac{1}{2}\left[\frac{8}{27}(x-1)^2+2(-\frac{2}{9}(x-1)(y-2)) + \frac{2}{3}(y-2)^2\right] \\
&\approx 3+\frac{1}{3}(x-1)+2(y-2)+\frac{4}{27}(x-1)^2-\frac{2}{9}(x-1)(y-2)+\frac{1}{3}(y-2)^2
\end{aligned}$$

gives the Taylor series up to second order for $f(x,y) = \sqrt{x^2+y^3}$ near $(1,2)$.

$$\begin{aligned}
\sqrt{(1.02)^2+(1.97)^3} &\approx 3+\frac{1}{3}(0.02)+2(-0.03)+\frac{4}{27}(0.02)^2-\frac{2}{9}(0.02)(-0.03)+\frac{1}{3}(-0.03)^2 \\
&\approx 2.947159259
\end{aligned}$$

Example 4: Calculate the second degree Taylor series approximation for $f(x,y) = \cos x \sin y$ near $(\pi, \frac{\pi}{2})$.

We need to calculate all of the derivatives up to 2nd order at $(x_0, y_0) = (\pi, \frac{\pi}{2})$.

Taylor Series for Functions of Two Variables

$$\begin{aligned}
f(x,y) &= \cos x \sin y & f(\pi,\tfrac{\pi}{2}) &= -1\\
f_x(x,y) &= -\sin x \sin y & f_x(\pi,\tfrac{\pi}{2}) &= 0\\
f_y(x,y) &= \cos x \cos y & f_y(\pi,\tfrac{\pi}{2}) &= 0\\
f_{xx}(x,y) &= -\cos x \sin y & f_{xx}(\pi,\tfrac{\pi}{2}) &= 1\\
f_{xy}(x,y) &= -\sin x \cos y & f_{xy}(\pi,\tfrac{\pi}{2}) &= 0\\
f_{yy}(x,y) &= -\cos x \sin y & f_{yy}(\pi,\tfrac{\pi}{2}) &= 1
\end{aligned}$$

And so

$$\cos x \sin y \approx -1 + \frac{1}{2}(x-\pi)^2 + \frac{1}{2}(y-\frac{\pi}{2})^2$$

Exercises

1. Find the Taylor series for the following functions:

(a). $f(x,y) = \dfrac{1}{3+x^2y}$ near $(0,0)$.

(b) $f(x,y) = \arctan(xy-y)$ near $(1,0)$.

(c) $f(x,y) = e^{x^2+y^2}$ near $(0,0)$.

2. Find the terms up to degree three for the Taylor series of $f(x,y) = \dfrac{1}{3+2x-y}$ near $(1,2)$.

3. Use the Taylor series to degree two to approximate $\dfrac{(2.98)^2}{(2.97)^2}$ using $f(x,y) = \dfrac{x^2}{y^2}$.