Taylor Series for Functions of Two Variables - Exercise Solutions

1. (a) $f(x,y) = \frac{1}{3+x^2y} = \frac{1/3}{1-(-\frac{1}{3}x^2y)} = \sum_{n=0}^{\infty} \frac{1}{3}\left(-\frac{1}{3}x^2y\right)^n$

(geometric power series) $= \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} x^{2n} y^n$

(b) $f(x,y) = \arctan(xy-y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} (xy-y)^{2n+1}$

(using power series for arctan) $= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \left[(x-1)y\right]^{2n+1}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} (x-1)^{2n+1} y^{2n+1}$

(c) $f(x,y) = e^{x^2+y^2} = \sum_{n=0}^{\infty} \frac{1}{n!}(x^2+y^2)^n$ (using power series for exp)

$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{r=0}^{n} \binom{n}{r} (x^2)^r (y^2)^{n-r} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{r=0}^{n} \binom{n}{r} x^{2r} y^{2(n-r)}$

2. $f(x,y) = \frac{1}{3+2x-y}$ $\qquad f(1,2) = \frac{1}{3}$

$f_x(x,y) = \frac{-2}{(3+2x-y)^2}$ $\qquad f_x(1,2) = \frac{-2}{9}$

$f_y(x,y) = \frac{1}{(3+2x-y)^2}$ $\qquad f_y(1,2) = \frac{1}{9}$

$f_{xx}(x,y) = \frac{8}{(3+2x-y)^3}$ $\qquad f_{xx}(1,2) = \frac{8}{27}$

$f_{xy}(x,y) = \frac{-4}{(3+2x-y)^3}$ $\qquad f_{xy}(1,2) = \frac{-4}{27}$

$f_{yy}(x,y) = \frac{2}{(3+2x-y)^3}$ $\qquad f_{yy}(1,2) = \frac{2}{27}$

$f_{xxx}(x,y) = \frac{-48}{(3+2x-y)^4}$ $\qquad f_{xxx}(1,2) = \frac{-48}{81} = \frac{-16}{27}$

$f_{xxy}(x,y) = \frac{24}{(3+2x-y)^4}$ $\qquad f_{xxy}(1,2) = \frac{24}{81} = \frac{8}{27}$

$f_{xyy}(x,y) = \frac{-12}{(3+2x-y)^4}$ $\qquad f_{xyy}(1,2) = \frac{-12}{81} = \frac{-4}{27}$

$f_{yyy}(x,y) = \frac{6}{(3+2x-y)^4}$ $\qquad f_{yyy}(1,2) = \frac{6}{81} = \frac{2}{27}$

$$\therefore f(x,y) \approx \frac{1}{3} - \frac{2}{9}(x-1) + \frac{1}{9}(y-2) + \frac{1}{2!}\left[\frac{8}{27}(x-1)^2 + 2\left(\frac{-4}{27}\right)(x-1)(y-2) + \frac{2}{27}(y-2)^2\right]$$

$$+ \frac{1}{3!}\left[\frac{-16}{27}(x-1)^3 + 3\left(\frac{8}{27}\right)(x-1)^2(y-2) + 3\left(\frac{-4}{27}\right)(x-1)(y-2)^2 + \frac{2}{27}(y-2)^3\right]$$

$$= \frac{1}{3} - \frac{2}{9}(x-1) + \frac{1}{9}(y-2) + \frac{4}{27}(x-1)^2 - \frac{4}{27}(x-1)(y-2) + \frac{1}{27}(y-2)^2$$

$$-\frac{8}{81}(x-1)^3 + \frac{4}{27}(x-1)^2(y-2) - \frac{2}{27}(x-1)(y-2)^2 + \frac{1}{81}(y-2)^3.$$

OR $\quad f(x,y) = \dfrac{1}{3-[-2(x-1)+(y-2)]} = \dfrac{1/3}{1-\frac{1}{3}[-2(x-1)+(y-2)]}$

$$= \sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{1}{3}[-2(x-1)+(y-2)]\right)^n = \sum_{n=0}^{\infty} \frac{1}{3^{n+1}}\left[-2(x-1)+(y-2)\right]^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{3^{n+1}} \sum_{r=0}^{n} \binom{n}{r} (-2)^r (x-1)^r (y-2)^{n-r}$$

$$\approx \frac{1}{3} + \frac{1}{9}\left[(-2)(x-1)+(y-2)\right] + \frac{1}{27}\left[(-2)^2(x-1)^2 + 2(-2)(x-1)(y-2)\right.$$

$$\left. + (y-2)^2\right] + \frac{1}{81}\left[(-2)^3(x-1)^3 + 3(-2)^2(x-1)^2(y-2)\right.$$

$$\left. + 3(-2)(x-1)(y-2)^2 + (y-2)^3\right]$$

$$= \frac{1}{3} - \frac{2}{9}(x-1) + \frac{1}{9}(y-2) + \frac{4}{27}(x-1)^2 - \frac{4}{27}(x-1)(y-2) + \frac{1}{27}(y-2)^2$$

$$- \frac{8}{81}(x-1)^3 + \frac{4}{27}(x-1)^2(y-2) - \frac{2}{27}(x-1)(y-2)^2 + \frac{1}{81}(y-2)^3.$$

3. Using Taylor Series near $(3,3)$.

$f(x,y) = \frac{x^2}{y^2}$ $\qquad f(3,3) = 1$

$f_x(x,y) = \frac{2x}{y^2}$ $\qquad f_x(3,3) = \frac{2}{3}$

$f_y(x,y) = -\frac{2x^2}{y^3}$ $\qquad f_y(3,3) = -\frac{2}{3}$

$f_{xx}(x,y) = \frac{2}{y^2}$ $\qquad f_{xx}(3,3) = \frac{2}{9}$

$f_{xy}(x,y) = \frac{-4x}{y^3}$ $\qquad f_{xy}(3,3) = -\frac{4}{9}$

$f_{yy}(x,y) = \frac{6x^2}{y^4}$ $\qquad f_{yy}(3,3) = \frac{2}{3}$

$$\therefore f(x,y) \approx 1 + \frac{2}{3}(x-3) - \frac{2}{3}(y-3) + \frac{1}{2!}\left[\frac{2}{9}(x-3)^2 + 2\left(-\frac{4}{9}\right)(x-3)(y-3) + \frac{2}{3}(y-3)^2\right]$$

$$= 1 + \frac{2}{3}(x-3) - \frac{2}{3}(y-3) + \frac{1}{9}(x-3)^2 - \frac{4}{9}(x-3)(y-3) + \frac{1}{3}(y-3)^2$$

$$\therefore \frac{(2.98)^2}{(2.97)^2} = f(2.98, 2.97) \approx 1 + \frac{2}{3}(-0.02) - \frac{2}{3}(-0.03) + \frac{1}{9}(-0.02)^2 - 4/9(-0.02)(-0.03) + \frac{1}{3}(-0.03)^2$$

$$\approx 1.0067\overline{4} \qquad \left(\text{or } \frac{90607}{90000}\right)$$

(compare with calculator value of $\frac{(2.98)^2}{(2.97)^2} \approx 1.006745343$)