Green's Theorem

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Let R be a simply connected region with a piecewise smooth boundary C, oriented counterclockwise (i.e. C is traversed *once* so that the region R always lies to the *left*). If M and N have continuous first partial derivatives in an open region containing R, then

$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA.$$

Note that Green's theorem only applies to 2D vector fields, $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$, whereby

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA.$$

Another form of Green's Theorem

Recall $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$ integrates the tangent component of $\mathbf{F} = \langle M, N \rangle$ around the curve C, where $\mathbf{T} = \mathbf{r}'(s) = \left\langle \frac{dx}{ds}, \frac{dy}{ds} \right\rangle$. Suppose we instead want to integrate the (outward) normal component $\mathbf{N}$ of $\mathbf{F}$ around C. Then $\mathbf{N} = \left\langle \frac{dy}{ds}, -\frac{dx}{ds} \right\rangle$, and

$$\begin{aligned}
\int_C \mathbf{F} \cdot \mathbf{N}\, ds &= \int_C \langle M, N \rangle \cdot \left\langle \frac{dy}{ds}, -\frac{dx}{ds} \right\rangle ds = \int_C \left(M\frac{dy}{ds} - N\frac{dx}{ds} \right) ds = \int_C M dy - N dx \\
&= \int_C (-N) dx + M dy = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA \quad \text{(by Green's Theorem)} \\
&= \iint_R \operatorname{div} \mathbf{F}\, dA \qquad \text{or} \qquad \iint_R \boldsymbol{\nabla} \cdot \mathbf{F}\, dA
\end{aligned}$$

Another form of Green's theorem is therefore

$$\int_C \mathbf{F} \cdot \mathbf{N}\, ds = \iint_R \boldsymbol{\nabla} \cdot \mathbf{F}\, dA.$$

An extension of this form to three dimensions is called the Divergence Theorem.