Extrema

Extrema = Extreme Values = Maxima or Minima = Maximum or Minimum Values
Extremum = Extreme Value = Maximum or Minimum = Maximum or Minimum Value

Definition of Extrema (or Absolute Extrema or Global Extrema)

Let f be a function defined on a region R containing (x_0, y_0).

1. $f(x_0, y_0)$ is the **minimum** (or **absolute minimum** or **global minimum**) **of** f **on** R if $f(x_0, y_0) \leq f(x, y)$ for all (x, y) in R.
2. $f(x_0, y_0)$ is the **maximum** (or **absolute maximum** or **global maximum**) **of** f **on** R if $f(x_0, y_0) \geq f(x, y)$ for all (x, y) in R.

Definition of Relative Extrema (or Local Extrema)

Let f be a function defined on a region R containing (x_0, y_0).

1. f has a **relative minimum** (or **local minimum**) at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$ for all (x, y) in an open disk containing (i.e. a δ-neighbourhood of) (x_0, y_0).
2. f has a **relative maximum** (or **local maximum**) if $f(x_0, y_0) \geq f(x, y)$ for all (x, y) in an open disk containing (i.e. a δ-neighbourhood of) (x_0, y_0).

Definition of Critical Point

Let f be a function defined on an open region R containing (x_0, y_0). The point (x_0, y_0) is a **critical point** of f if one of the following is true.

1. $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ (i.e. $\boldsymbol{\nabla} f(x_0, y_0) = \mathbf{0}$)
2. $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

Theorem 13.15: **Extreme Value Theorem**

Let f be a continuous function of two variables x and y defined on a closed, bounded region R in the xy-plane.

1. There is at least one point in R at which f takes on a minimum value.
2. There is at least one point in R at which f takes on a maximum value.

Theorem 13.16: **Relative Extrema Occur only at Critical Points**

If f has a relative extremum at (x_0, y_0) on an open region R, then (x_0, y_0) is a critical point of f.

Theorem 13.17: **Second Partials Test**

Let f have continuous second partial derivatives on an open region containing a point (a, b) for which

$$f_x(a, b) = 0 \qquad \text{and} \qquad f_y(a, b) = 0.$$

To test for relative extrema of f, consider the quantity

$$d = \begin{vmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{vmatrix} = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

1. If $d > 0$ and $f_{xx}(a, b) > 0$, then f has a **relative minimum** at (a, b).
2. If $d > 0$ and $f_{xx}(a, b) < 0$, then f has a **relative maximum** at (a, b).
3. If $d < 0$, then $(a, b, f(a, b))$ is a **saddle point**.
4. If $d = 0$, then the test is inconclusive.