Conservative Vector Fields

Theorem 15.5(a): Fundamental Theorem of Line Integrals (2D)
Let C be a piecewise smooth curve lying in an open region R and given by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j},\ a \leq t \leq b.$$

If $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$ is conservative in R, and M and N are continuous in R, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \boldsymbol{\nabla} f \cdot d\mathbf{r} = f(x(b), y(b)) - f(x(a), y(a)),$$

where f is a potential function of $\mathbf{F}$, i.e. $\mathbf{F}(x, y) = \boldsymbol{\nabla} f(x, y)$.

Theorem 15.5(b): Fundamental Theorem of Line Integrals (3D)
Let C be a piecewise smooth curve lying in an open region Q and given by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},\ a \leq t \leq b.$$

If $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative in Q, and M, N and P are continuous in Q, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \boldsymbol{\nabla} f \cdot d\mathbf{r} = f(x(b), y(b), z(b)) - f(x(a), y(a), z(a)),$$

where f is a potential function of $\mathbf{F}$, i.e. $\mathbf{F}(x, y, z) = \boldsymbol{\nabla} f(x, y, z)$.

Theorem 15.6: Independence of Path and Conservative Vector Fields
If $\mathbf{F}$ is continuous on an open connected region, then the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path if and only if $\mathbf{F}$ is conservative.

Theorem 15.7: Equivalent Conditions
Let $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$ in 2D or $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ in 3D have continuous first partial derivatives in an open connected region R, and let C be a piecewise smooth curve in R. Then the following are equivalent.

1. $\mathbf{F}$ is conservative, i.e. $\mathbf{F} = \boldsymbol{\nabla} f$ for some function f.
2. $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path.
3. $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every *closed* curve C in R.