Binormal Example

Exercise: Find $\mathbf{T}(t)$, $\mathbf{N}(t)$ and $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ for the curve $\mathbf{r}(t) = \langle \cos t, \sin t, \cos t \rangle$, $0 \le t \le 2\pi$.

Solution: Differentiating $\mathbf{r}(t)$ gives

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, -\sin t \rangle,$$

for which

$$\|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + \sin^2 t} = \sqrt{1 + \sin^2 t}.$$

Therefore,

$$\mathbf{T}(t) = \frac{\langle -\sin t, \cos t, -\sin t \rangle}{\sqrt{1 + \sin^2 t}}.$$

Applying the quotient rule to the components of $\mathbf{T}(t)$ gives

$$\begin{aligned} \frac{d}{dt} \frac{-\sin t}{\sqrt{1+\sin^2 t}} &= \frac{\sqrt{1+\sin^2 t}(-\cos t) - (-\sin t)(1/2)(1+\sin^2 t)^{-1/2}(2 \sin t \cos t)}{1 + \sin^2 t} \\ &= \frac{(1+\sin^2 t)^{-1/2}(-\cos t)((1+\sin^2 t) - \sin^2 t)}{1+\sin^2 t} = \frac{-\cos t}{(1+\sin^2 t)^{3/2}} \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt} \frac{\cos t}{\sqrt{1+\sin^2 t}} &= \frac{\sqrt{1+\sin^2 t}(-\sin t) - (\cos t)(1/2)(1+\sin^2 t)^{-1/2}(2 \sin t \cos t)}{1 + \sin^2 t} \\ &= \frac{(1+\sin^2 t)^{-1/2}(-\sin t)((1+\sin^2 t) + \cos^2 t)}{1+\sin^2 t} = \frac{-2\sin t}{(1+\sin^2 t)^{3/2}}. \end{aligned}$$

Therefore

$$\mathbf{T}'(t) = \frac{\langle -\cos t, -2\sin t, -\cos t \rangle}{(1+\sin^2 t)^{3/2}}.$$

To normalize $\mathbf{T}'(t)$, we can focus on its direction $\langle -\cos t, -2\sin t, -\cos t \rangle$ and disregard the scalar $1/(1+\sin^2 t)^{3/2}$, since that factor would later cancel. Since

$$\|\langle -\cos t, -2\sin t, -\cos t \rangle\| = \sqrt{\cos^2 t + 4\sin^2 t + \cos^2 t} = \sqrt{2 + 2\sin^2 t} = \sqrt{2}\sqrt{1+\sin^2 t},$$

then

$$\mathbf{N}(t) = \frac{\langle -\cos t, -2\sin t, -\cos t \rangle}{\sqrt{2}\sqrt{1+\sin^2 t}}.$$

To find the binormal vector $\mathbf{B}(t)$ we compute the cross product

$$\begin{aligned}
\mathbf{B}(t) &= \mathbf{T}(t) \times \mathbf{N}(t) \\
&= \frac{\langle -\sin t, \cos t, -\sin t\rangle}{\sqrt{1+\sin^2 t}} \times \frac{\langle -\cos t, -2\sin t, -\cos t\rangle}{\sqrt{2}\sqrt{1+\sin^2 t}} \\
&= \frac{1}{\sqrt{2}(1+\sin^2 t)} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & -\sin t \\ -\cos t & -2\sin t & -\cos t \end{vmatrix} \\
&= \frac{1}{\sqrt{2}(1+\sin^2 t)} \langle -\cos^2 t - 2\sin^2 t, -\sin t\cos t + \sin t \cos t, 2\sin^2 t + \cos^2 t\rangle \\
&= \frac{1}{\sqrt{2}(1+\sin^2 t)} \langle -1 + \sin^2 t - 2\sin^2 t, 0, 2\sin^2 t + 1 - \sin^2 t\rangle \\
&= \frac{1}{\sqrt{2}(1+\sin^2 t)} \langle -1 - \sin^2 t, 0, 1 + \sin^2 t\rangle \\
&= \left\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle.
\end{aligned}$$

Note that in this example, $\mathbf{B}(t)$ is constant, which means $\mathbf{r}(t)$ must be a plane curve. The plane containing the curve has normal vector $\mathbf{n} = \langle -1, 0, 1\rangle$ in the direction of $\mathbf{B}(t)$. The curve itself happens to be an ellipse formed by intersecting the unit cylinder $x^2 + y^2 = 1$ with the plane $z = x$.