Partial Fractions

Rational functions can be added (or subtracted) by finding the least common denominator. For example,

$$\frac{2}{x-3} + \frac{5}{x+2} = \frac{2(x+2) + 5(x-3)}{(x-3)(x+2)} = \frac{7x-11}{x^2-x-6}.$$

Partial fraction decomposition is the algebraic process of reversing the above operation, by starting with

$$\frac{7x-11}{x^2-x-6}$$

and converting it into the sum of

$$\frac{2}{x-3} \text{ and } \frac{5}{x+2},$$

which are called **partial fractions**. This can make integration easier. It follows that

$$\int \frac{7x-11}{x^2-x-6}\,dx = \int \left(\frac{2}{x-3} + \frac{5}{x+2}\right) dx = 2\ln|x-3| + 5\ln|x+2| + C.$$

In order to decompose a rational function into partial fractions, the rational function must be proper – the degree of the numerator polynomial must be less than the degree of the denominator polynomial. If it's not, then long division should be used to convert the improper rational function into the sum of a polynomial and a proper rational function. For example,

$$\frac{-3x^3 + 3x^2 + 25x - 11}{x^2-x-6} = -3x + \frac{7x-11}{x^2-x-6}.$$

Given a proper rational function of the form $N(x)/D(x)$, where $N(x)$ and $D(x)$ are the numerator and denominator polynomials, respectively, $D(x)$ must be completely factored over the reals into a product of linear factors of the form $px+q$ and irreducible quadratic factors of the form ax^2+bx+c (where $b^2-4ac<0$). Factoring the numerator is generally not necessary unless you suspect that the numerator and denominator have a common factor, which could then be canceled, allowing you to simplify the rational function by reducing it to lowest terms. In the example above,

$$\frac{7x-11}{x^2-x-6} = \frac{7x-11}{(x-3)(x+2)},$$

where the denominator has been factored into the two linear factors $x-3$ and $x+2$.

In some cases, factoring $D(x)$ produces repeated factors. For example,

$$\frac{x^5+x^3-6x^2-4}{x^7+2x^5+x^3} = \frac{x^5+x^3-6x^2-4}{x^3(x^2+1)^2},$$

where the linear factor x is repeated three times and the irreducible quadratic factor x^2+1 is repeated twice.

The factors of $D(x)$ dictate the number and nature of the partial fractions in the partial fraction decomposition of the rational function. Each linear factor $px+q$, if not repeated, generates a partial fraction of the form

$$\frac{A}{px+q},$$

and each irreducible quadratic factor ax^2+bx+c, if not repeated, generates a partial fraction of the form

$$\frac{Bx+C}{ax^2+bx+c},$$

where A, B and C are unknown constants to be determined. For example, the partial fraction decomposition of

$$\frac{7x-11}{(x-3)(x+2)}$$

must have the form

$$\frac{7x-11}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2},$$

for some constants A and B. Multiplying both sides of this equation by the least common denominator $(x-3)(x+2)$ produces

$$7x-11 = A(x+2) + B(x-3),$$

which is known as the **basic equation**.

In the event of repeated linear factors $(px+q)^m$, there will be m partial fractions of the form

$$\frac{A_1}{px+q} + \frac{A_2}{(px+q)^2} + \frac{A_3}{(px+q)^3} + \cdots + \frac{A_m}{(px+q)^m}.$$

Similarly, in the event of repeated irreducible quadratic factors $(ax^2+bx+c)^n$, there will be n partial fractions of the form

$$\frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \frac{B_3x+C_3}{(ax^2+bx+c)^3} + \cdots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}.$$

For example, the partial fraction decomposition of

$$\frac{x^5+x^3-6x^2-4}{x^3(x^2+1)^2},$$

must have the form

$$\frac{x^5+x^3-6x^2-4}{x^3(x^2+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{(x^2+1)^2},$$

for some constants $A, B, C, \ldots, G$. Multiplying both sides of this equation by the least common denominator $x^3(x^2+1)^2$ produces the basic equation

$$x^5+x^3-6x^2-4 = Ax^2(x^2+1)^2 + Bx(x^2+1)^2 + C(x^2+1)^2 + (Dx+E)x^3(x^2+1) + (Fx+G)x^3.$$

Once you know the proper form of the partial fraction decomposition, all that remains is to solve for the unknown constants, A, B, C, etc. This is done using the basic equation. There are two ways in which the basic equation can be used to find these constants. The first is to note that the equation must be satisfied by *every* real number x, including (perhaps surprisingly) values that are not in the domain of the original rational function because they would cause division by zero. Plugging various x values into the basic equation produces linear equations involving the unknown constants that can then be used to solve for the constants. In the case of linear factors, the most convenient choices of x values are often precisely those that make the particular factors zero. Other values of x that are often nice to work with are $0, \pm 1, \pm 2$, etc.

Consider the basic equation

$$7x - 11 = A(x+2) + B(x-3).$$

Letting $x = 3$ in this equation results in

$$10 = A(5) + B(0),$$

from which you get $A = 2$. Likewise, letting $x = -2$ in the same equation results in

$$-25 = A(0) + B(-5),$$

from which you get $B = 5$. The partial fraction decomposition of

$$\frac{7x-11}{x^2-x-6}$$

is therefore

$$\frac{7x-11}{x^2-x-6} = \frac{2}{x-3} + \frac{5}{x+2}.$$

Next consider the basic equation

$$x^5 + x^3 - 6x^2 - 4 = Ax^2(x^2+1)^2 + Bx(x^2+1)^2 + C(x^2+1)^2 + (Dx+E)x^3(x^2+1) + (Fx+G)x^3.$$

Here, the most convenient value for x is 0. Letting $x = 0$ eliminates all the terms except -4 on the left side and the C term on the right side, which allows you to solve for C to get $C = -4$. Other values of x produce linear equations in terms of these constants. For example, if $x = 1$, then

$$-8 = 4A + 4B + 4C + 2D + 2E + F + G,$$

while letting $x = -1$ gives

$$-12 = 4A - 4B + 4C + 2D - 2E + F - G,$$

and letting $x = 2$ gives

$$12 = 100A + 50B + 25C + 80D + 40E + 16F + 8G.$$

Given that there are 7 unknowns (assuming you hadn't already found C), you can plug 7 different values of x into the basic equation and produce a system of 7 linear equations. Any of various linear algebra techniques can then be used to solve for $A, B, C, \ldots, G$.

A second approach to using the basic equation to create linear equations relating the unknown constants comes from the observation that the polynomial functions of x on either side of the basic equation must be the same, and so in particular they must have the same coefficients of x. Expanding the right side of

$$x^5 + x^3 - 6x^2 - 4 = Ax^2(x^2+1)^2 + Bx(x^2+1)^2 + C(x^2+1)^2 + (Dx+E)x^3(x^2+1) + (Fx+G)x^3$$

and combining like terms gives

$$\begin{aligned}
&x^5 + x^3 - 6x^2 - 4 \\
&= A(x^6 + 2x^4 + x^2) + B(x^5 + 2x^3 + x) + C(x^4 + 2x^2 + 1) + D(x^6 + x^4) + E(x^5 + x^3) + Fx^4 + Gx^3 \\
&= (A+D)x^6 + (B+E)x^5 + (2A + C + D + F)x^4 + (2B + E + G)x^3 + (A + 2C)x^2 + Bx + C.
\end{aligned}$$

Equating the coefficients of like terms on opposite sides of the basic equation produces linear equations relating the unknowns that can be used instead of, or in addition to, equations found by plugging x values into the basic equation. In this example, you get the following equations:

$$\begin{aligned}
0 &= A + D && \text{coefficient of } x^6 \\
1 &= B + E && \text{coefficient of } x^5 \\
0 &= 2A + C + D + F && \text{coefficient of } x^4 \\
1 &= 2B + E + G && \text{coefficient of } x^3 \\
-6 &= A + 2C && \text{coefficient of } x^2 \\
0 &= B && \text{coefficient of } x \\
-4 &= C && \text{constant term.}
\end{aligned}$$

From this system of equations it is easy to solve for $A, B, \ldots, G$ and obtain

$$A = 2, \quad B = 0, \quad C = -4, \quad D = -2, \quad E = 1, \quad F = 2, \quad G = 0.$$

The partial fraction decomposition of

$$\frac{x^5 + x^3 - 6x^2 - 4}{x^3(x^2+1)^2}$$

is therefore

$$\begin{aligned}
\frac{x^5 + x^3 - 6x^2 - 4}{x^3(x^2+1)^2} &= \frac{2}{x} + \frac{0}{x^2} + \frac{-4}{x^3} + \frac{-2x+1}{x^2+1} + \frac{2x+0}{(x^2+1)^2} \\
&= \frac{2}{x} - \frac{4}{x^3} + \frac{-2x+1}{x^2+1} + \frac{2x}{(x^2+1)^2}.
\end{aligned}$$

Finally, integrating becomes relatively straightforward:

$$\begin{aligned}
\int \frac{x^5 + x^3 - 6x^2 - 4}{x^7 + 2x^5 + x^3}\, dx &= \int \frac{x^5 + x^3 - 6x^2 - 4}{x^3(x^2+1)^2}\, dx \\
&= \int \left(\frac{2}{x} - \frac{4}{x^3} + \frac{-2x+1}{x^2+1} + \frac{2x}{(x^2+1)^2} \right) dx \\
&= 2\int \frac{1}{x}\, dx - 4\int \frac{1}{x^3}\, dx - \int \frac{2x}{x^2+1}\, dx + \int \frac{1}{x^2+1}\, dx + \int \frac{2x}{(x^2+1)^2}\, dx \\
&= 2\ln|x| + \frac{2}{x^2} - \ln(x^2+1) + \arctan(x) - \frac{1}{x^2+1} + C,
\end{aligned}$$

where the $+C$ integration constant is not to be confused with the constant $C = -4$ associated with the partial fraction $\dfrac{C}{x^3}$.