CAMOSUN

Name: SOLUTIONS

Mark: ___/25

MATH 101
Assignment 4

1. (2 marks) Use formulas from the Integration Tables to assist you in finding the indefinite integral

$$\int (\ln x)^3 \, dx.$$

See Appendix B in the textbook for the complete list of 100 integration formulas. Identify, by number, which formula(s) you use and simplify your answer.

Note: On this assignment and subsequent assignments and tests, you may only use formulas from the Integration Tables if a question, like this one, explicitly instructs you to do so. You would otherwise be expected to show all your integration work.

$$\int (\ln x)^3 dx \overset{\#91}{=} x(\ln x)^3 - 3\int (\ln x)^2 dx$$

$$\overset{\#90}{=} x(\ln x)^3 - 3x\left[2 - 2\ln x + (\ln x)^2\right] + C$$

$$= x(\ln x)^3 - 6x + 6x\ln x - 3x(\ln x)^2 + C$$

2. Use a trigonometric substitution to integrate.

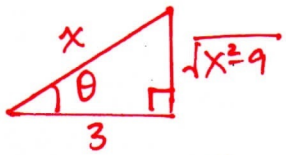
(a) (3 marks) $\int \frac{3x^3}{\sqrt{x^2-9}}\,dx$

$= \int \frac{3(3\sec\theta)^3}{3\tan\theta} \cdot 3\sec\theta\tan\theta\,d\theta$

Let $x = 3\sec\theta$

then $dx = 3\sec\theta\tan\theta\,d\theta$

$\sec\theta = \frac{x}{3}$



$\tan\theta = \frac{\sqrt{x^2-9}}{3}$

$\sqrt{x^2-9} = 3\tan\theta$

$= 81\int \sec^4\theta\,d\theta$

$= 81\int (\tan^2\theta+1)\sec^2\theta\,d\theta$

$= 81\left[\frac{1}{3}\tan^3\theta + \tan\theta\right] + C$

$= 81\left[\frac{1}{3}\left(\frac{\sqrt{x^2-9}}{3}\right)^3 + \frac{\sqrt{x^2-9}}{3}\right] + C$

$= (x^2-9)^{3/2} + 27\sqrt{x^2-9} + C$

or $\sqrt{x^2-9}\,(x^2+18) + C$ if factored

(b) (3 marks) $\int \sqrt{4x-x^2}\,dx$ $= \int \sqrt{4-(x^2-4x+4)}\,dx = \int \sqrt{4-(x-2)^2}\,dx$

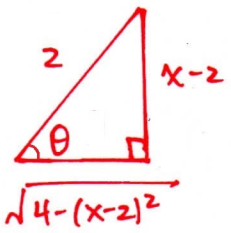
Let $x-2 = 2\sin\theta$

then $x = 2\sin\theta + 2$

and $dx = 2\cos\theta\,d\theta$.

$\sin\theta = \frac{x-2}{2}$

and $\theta = \arcsin\frac{x-2}{2}$



$\cos\theta = \frac{\sqrt{4-(x-2)^2}}{2}$ or $\frac{\sqrt{4x-x^2}}{2}$

and $\sqrt{4-(x-2)^2} = 2\cos\theta$

$= \int 2\cos\theta \cdot 2\cos\theta\,d\theta$

$= 4\int \cos^2\theta\,d\theta$

$= 4\int \frac{1+\cos 2\theta}{2}\,d\theta$

$= 2\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$

$= 2\left[\theta + \sin\theta\cos\theta\right] + C$

$= 2\left[\arcsin\frac{x-2}{2} + \frac{x-2}{2}\cdot\frac{\sqrt{4x-x^2}}{2}\right] + C$

$= 2\arcsin\frac{x-2}{2} + \frac{1}{2}(x-2)\sqrt{4x-x^2} + C$

#2(b) Alternate solution...

$$I = \int \sqrt{4x-x^2}\,dx = \int \sqrt{x}\sqrt{4-x}\,dx = \int \sqrt{x}\sqrt{4-(\sqrt{x})^2}\,dx$$

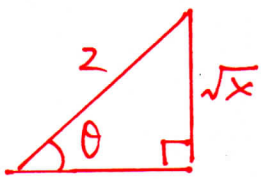
Let $\sqrt{x} = 2\sin\theta$

$x = 4\sin^2\theta$

$dx = 8\sin\theta\cos\theta\,d\theta$

$\sin\theta = \frac{\sqrt{x}}{2}$

$\theta = \arcsin\frac{\sqrt{x}}{2}$



$\sqrt{4-x} = 2\cos\theta$

$$I = \int (2\sin\theta)(2\cos\theta)(8\sin\theta\cos\theta)\,d\theta = 32\int \sin^2\theta\cos^2\theta\,d\theta$$

$$= 32\int \left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{1+\cos 2\theta}{2}\right)d\theta = 8\int (1-\cos^2 2\theta)\,d\theta$$

$$= 8\int \left(1-\frac{1+\cos 4\theta}{2}\right)d\theta = 4\int (1-\cos 4\theta)\,d\theta$$

$$= 4\theta - \sin 4\theta + C = 4\theta - 2\sin 2\theta\cos 2\theta + C$$

$$= 4\theta - 2(2\sin\theta\cos\theta)(1-2\sin^2\theta) + C$$

$$= 4\arcsin\frac{\sqrt{x}}{2} - 4\left(\frac{\sqrt{x}}{2}\right)\left(\frac{\sqrt{4-x}}{2}\right)\left(1-2\left(\frac{\sqrt{x}}{2}\right)^2\right) + C$$

$$= 4\arcsin\frac{\sqrt{x}}{2} - \sqrt{4x-x^2}\left(1-\frac{x}{2}\right) + C$$

$$= 4\arcsin\frac{\sqrt{x}}{2} + \frac{1}{2}(x-2)\sqrt{4x-x^2} + C$$

Note: $4\arcsin\frac{\sqrt{x}}{2} = 2\arcsin\frac{x-2}{2} + \pi$

∴ the "+C" differs by π in these 2 solutions

3. Use partial fractions to integrate.

(a) (3 marks) $\int \frac{2x+9}{x^3-6x^2+9x}\,dx$ $= \int \frac{2x+9}{x(x-3)^2}\,dx$

$\frac{2x+9}{x(x-3)^2} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$

$2x+9 = A(x-3)^2 + Bx(x-3) + Cx$

$x=0 \Rightarrow 9 = 9A \Rightarrow A = 1$

$x=3 \Rightarrow 15 = 3C \Rightarrow C = 5$

$x=1 \Rightarrow 11 = 4A - 2B + C$

$11 = 4 - 2B + 5$

$2 = -2B \Rightarrow B = -1$

$= \int \left(\frac{1}{x} - \frac{1}{x-3} + \frac{5}{(x-3)^2}\right) dx$

$= \ln|x| - \ln|x-3| - \frac{5}{x-3} + C$

(b) (3 marks) $\int \frac{18x+9}{x^4+9x^2}\,dx$ $= \int \frac{18x+9}{x^2(x^2+9)}\,dx$

$\frac{18x+9}{x^2(x^2+9)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+9}$

$18x+9 = Ax(x^2+9) + B(x^2+9) + (Cx+D)x^2$

$= Ax^3 + 9Ax + Bx^2 + 9B + Cx^3 + Dx^2$

$= (A+C)x^3 + (B+D)x^2 + 9Ax + 9B$

$\therefore A+C=0,\ B+D=0,\ 9A=18,\ 9B=9$

$\Rightarrow A=2,\ B=1,\ C=-2,\ D=-1$

$= \int \left(\frac{2}{x} + \frac{1}{x^2} + \frac{-2x-1}{x^2+9}\right) dx$

$= 2\ln|x| - \frac{1}{x} - \int \frac{2x}{x^2+9}\,dx - \int \frac{1}{x^2+9}\,dx$

$= 2\ln|x| - \frac{1}{x} - \ln(x^2+9) - \frac{1}{3}\arctan\frac{x}{3} + C$

4. Evaluate the limits. If they do not exist, then determine whether they are ∞ or $-\infty$ or neither.

(a) (2 marks) $\lim_{x\to 0} \dfrac{\ln(\sec x)}{x^2}$

$\frac{0}{0}$

$\overset{\text{L'HR}}{=} \lim_{x\to 0} \dfrac{\frac{\sec x \tan x}{\sec x}}{2x} = \lim_{x\to 0} \dfrac{\tan x}{2x}$

$\frac{0}{0}$

$\overset{\text{L'HR}}{=} \lim_{x\to 0} \dfrac{\sec^2 x}{2} = \dfrac{1}{2}$

(b) (2 marks) $\lim_{x\to 0^+} \left(\dfrac{3}{x} - \csc x\right) = \lim_{x\to 0^+} \left(\dfrac{3}{x} - \dfrac{1}{\sin x}\right) = \lim_{x\to 0^+} \left(\dfrac{3\sin x - x}{x \sin x}\right)$

$\infty - \infty$ $\qquad$ $\frac{0}{0}$

$\overset{\text{L'HR}}{=} \lim_{x\to 0^+} \dfrac{3\cos x - 1}{x\cos x + \sin x} = \dfrac{2}{0^+} = \infty$ (d.n.e.)

(c) (2 marks) $\lim_{x\to\infty} \left(1 + 2\sin\dfrac{1}{x}\right)^x = L$ $\qquad$ 1^∞

$\ln L = \lim_{x\to\infty} \ln\left(1+2\sin\frac{1}{x}\right)^x = \lim_{x\to\infty} x \ln\left(1+2\sin\frac{1}{x}\right)$

$\infty \cdot 0$

$= \lim_{x\to\infty} \dfrac{\ln\left(1+2\sin\frac{1}{x}\right)}{\frac{1}{x}} \overset{\text{L'HR}}{=} \lim_{x\to\infty} \dfrac{\frac{1}{1+2\sin\frac{1}{x}} \cdot 2\cos\frac{1}{x} \cdot \left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}$

$\frac{0}{0}$

$= \lim_{x\to\infty} \dfrac{2\cos\frac{1}{x}}{1+2\sin\frac{1}{x}} = 2 \qquad \therefore \ L = e^2.$

5. Determine whether the improper integral converges or diverges and evaluate it if it converges. If it diverges, determine whether it diverges to ∞ or $-\infty$ or neither.

(a) (2 marks) $\int_0^\infty \sin^3 x \cos x \, dx$

$= \lim_{b\to\infty} \int_0^b \sin^3 x \cos x \, dx = \lim_{b\to\infty} \left[\frac{1}{4}\sin^4 x\right]_0^b$

$= \lim_{b\to\infty} \left[\frac{1}{4}\sin^4 b\right]$ d.n.e. since $\frac{1}{4}\sin^4 b$

oscillates between 0 and $\frac{1}{4}$

$\therefore$ integral diverges (but to neither ∞ nor $-\infty$)

(b) (3 marks) $\int_0^1 (\ln x + 2)\left(\frac{1}{2\sqrt{x}}\right) dx$ Infinite discontinuity at $x=0$.

Let $u = \ln x + 2$ and $dv = \frac{1}{2\sqrt{x}}\,dx$

then $du = \frac{1}{x}\,dx$ and $v = \sqrt{x}$

$\therefore \int (\ln x+2)\left(\frac{1}{2\sqrt{x}}\right)dx = (\ln x+2)\sqrt{x} - \int \frac{1}{\sqrt{x}}\,dx$

$= \sqrt{x}\ln x + 2\sqrt{x} - 2\sqrt{x} + C = \sqrt{x}\ln x + C$

$\therefore \int_0^1 (\ln x+2)\left(\frac{1}{2\sqrt{x}}\right)dx = \lim_{a\to 0^+} \int_a^1 (\ln x+2)\left(\frac{1}{2\sqrt{x}}\right)dx$

$= \lim_{a\to 0^+} \left[\sqrt{x}\ln x\right]_a^1 = \lim_{a\to 0^+} \left[-\sqrt{a}\ln a\right] = \lim_{a\to 0^+} \frac{-\ln a}{1/\sqrt{a}}$

$\overset{L'HR}{=} \lim_{a\to 0^+} \frac{-\frac{1}{a}}{-\frac{1}{2}a^{-3/2}} = \lim_{a\to 0^+} 2\sqrt{a} = 0$

$\therefore$ integral converges to 0