CAMOSUN

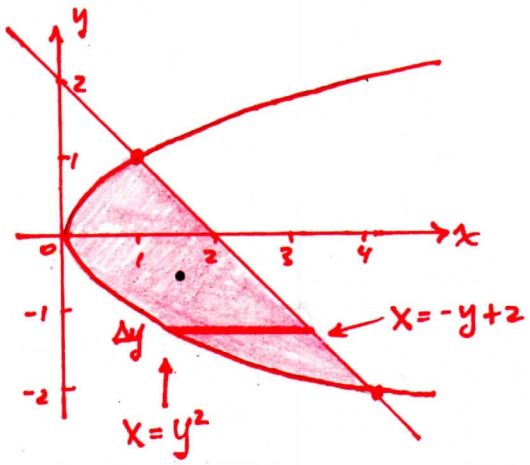
Name: SOLUTIONS

Mark: ___/25

MATH 101

Assignment 3

1. (5 marks) Find the center of mass $(\overline{x}, \overline{y})$ for the planar lamina having uniform density ρ bounded by the graphs of $x = -y + 2$ and $x = y^2$. Include a sketch of the region.



Intersection: $y^2 = -y+2$

$y^2+y-2=0$

$(y+2)(y-1)=0$

$y=-2$ or $y=1$

$\therefore$ points $(4,-2)$ and $(1,1)$

$$m = \rho\int_{-2}^{1}(-y+2-y^2)\,dy = \rho\left[-\tfrac{1}{2}y^2+2y-\tfrac{1}{3}y^3\right]_{-2}^{1} = \rho\left(\tfrac{7}{6}-\left(-\tfrac{10}{3}\right)\right) = \tfrac{9}{2}\rho$$

$$M_x = \rho\int_{-2}^{1} y(-y+2-y^2)\,dy = \rho\int_{-2}^{1}(-y^2+2y-y^3)\,dy = \rho\left[-\tfrac{1}{3}y^3+y^2-\tfrac{1}{4}y^4\right]_{-2}^{1}$$

$$= \rho\left(\tfrac{5}{12}-\tfrac{8}{3}\right) = -\tfrac{9}{4}\rho$$

$$M_y = \rho\int_{-2}^{1}\frac{y^2-y+2}{2}\cdot(-y+2-y^2)\,dy = \rho\int_{-2}^{1}\left(-\tfrac{1}{2}y^4+\tfrac{1}{2}y^2-2y+2\right)dy$$

$$= \rho\left[-\tfrac{1}{10}y^5+\tfrac{1}{6}y^3-y^2+2y\right]_{-2}^{1} = \rho\left(\tfrac{16}{15}-\left(-\tfrac{92}{15}\right)\right) = \tfrac{36}{5}\rho$$

$$\therefore (\overline{x},\overline{y}) = \left(\frac{M_y}{m},\frac{M_x}{m}\right) = \left(\frac{\frac{36}{5}\rho}{\frac{9}{2}\rho},\frac{-\frac{9}{4}\rho}{\frac{9}{2}\rho}\right) = \left(\tfrac{8}{5},-\tfrac{1}{2}\right)$$

2. (3 marks) Find the fluid force on the vertical triangular plate submerged in water, where the weight-density of water is 9,800 N/m^3.

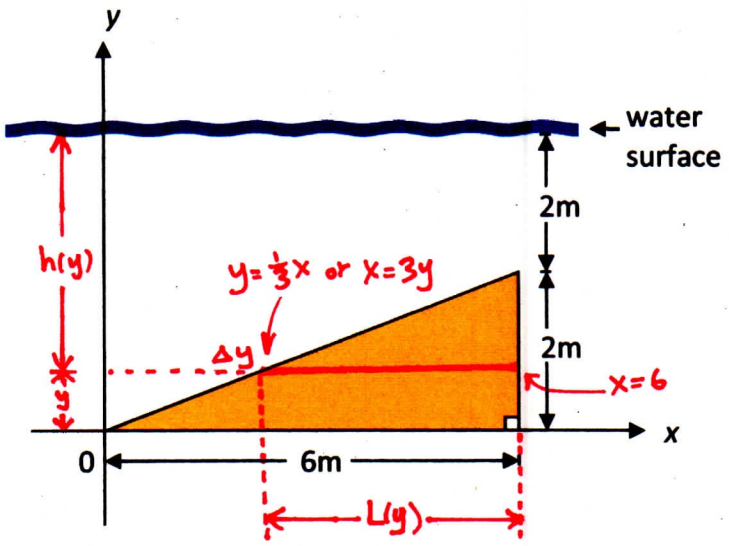
$w = 9800\ \text{N/m}^3$

$\Delta F = w\,h(y)\,L(y)\,\Delta y$

for $0 \le y \le 2$

where $h(y) = 4 - y$

and $L(y) = 6 - 3y$



$$\therefore F = w\int_0^2 (4-y)(6-3y)\,dy = w\int_0^2 (3y^2 - 18y + 24)\,dy$$

$$= w\left[y^3 - 9y^2 + 24y\right]_0^2 = 20w = 20(9800) = 196\,000\ \text{N}$$

3. (3 marks) Evaluate the definite integral by integrating by parts using the Table method.

$$\int_0^1 4x^4 e^{-2x}\,dx$$

	u	v'
+	$4x^4$	e^{-2x}
−	$16x^3$	$-\frac{1}{2}e^{-2x}$
+	$48x^2$	$\frac{1}{4}e^{-2x}$
−	$96x$	$-\frac{1}{8}e^{-2x}$
+	96	$\frac{1}{16}e^{-2x}$
−	0	$-\frac{1}{32}e^{-2x}$

$$= \left[-2x^4e^{-2x} - 4x^3e^{-2x} - 6x^2e^{-2x} - 6xe^{-2x} - 3e^{-2x}\right]_0^1$$

$$= \left[-e^{-2x}(2x^4 + 4x^3 + 6x^2 + 6x + 3)\right]_0^1$$

$$= -21e^{-2} + 3$$

4. Find the following integrals.

(a) (2 marks) $\int \frac{1}{1-e^x}\,dx$

$= \int \frac{1}{1-e^x} \cdot \frac{e^{-x}}{e^{-x}}\,dx$

$= \int \frac{e^{-x}}{e^{-x}-1}\,dx$ (Let $u = e^{-x}-1$, $du = -e^{-x}dx$)

$= \int \frac{-1}{u}\,du$

$= -\ln|u| + C$

$= -\ln|e^{-x}-1| + C$

OR

$= \int \frac{1-e^x+e^x}{1-e^x}\,dx$

$= \int \left(1 + \frac{e^x}{1-e^x}\right)dx$

$= x + \int \frac{e^x}{1-e^x}dx$ (Let $u = 1-e^x$, $du = -e^x dx$)

$= x + \int \frac{-1}{u}\,du$

$= x - \ln|u| + C$

$= x - \ln|1-e^x| + C$

(b) (2 marks) $\int \frac{1}{1-\sin\theta}\,d\theta$

$= \int \frac{1}{1-\sin\theta} \cdot \frac{1+\sin\theta}{1+\sin\theta}\,d\theta = \int \frac{1+\sin\theta}{1-\sin^2\theta}\,d\theta = \int \frac{1+\sin\theta}{\cos^2\theta}\,d\theta$

$= \int \left(\frac{1}{\cos^2\theta} + \frac{\sin\theta}{\cos^2\theta}\right)d\theta = \int (\sec^2\theta + \sec\theta\tan\theta)\,d\theta$

$= \tan\theta + \sec\theta + C.$

(c) (3 marks) $\int \underbrace{\sin 2x}_{u} \underbrace{\sinh 3x\, dx}_{dv}$ $= I$

	u	v'
+	$\sin 2x$	$\sinh 3x$
−	$2\cos 2x$	$\frac{1}{3}\cosh 3x$
+	$-4\sin 2x$	$\frac{1}{9}\sinh 3x$

$$I = \frac{1}{3}\sin 2x \cosh 3x - \frac{2}{9}\cos 2x \sinh 3x - \frac{4}{9}I$$

$$\frac{13}{9}I = \frac{1}{3}\sin 2x \cosh 3x - \frac{2}{9}\cos 2x \sinh 3x$$

$$I = \frac{3}{13}\sin 2x \cosh 3x - \frac{2}{13}\cos 2x \sinh 3x + C$$

(d) (2 marks) $\int \underbrace{\cosh^{-1} x}_{u}\, \underbrace{dx}_{dv}$

Let $u = \cosh^{-1} x$ $\quad dv = dx$

$du = \frac{1}{\sqrt{x^2-1}}dx$ $\quad v = x$

$$\int \cosh^{-1} x\, dx = x\cosh^{-1} x - \int \frac{x}{\sqrt{x^2-1}}dx$$

$$= x\cosh^{-1} x - \frac{1}{2}\int (x^2-1)^{-1/2}(2x)\,dx$$

$$= x\cosh^{-1} x - \sqrt{x^2-1} + C$$

(e) (2 marks) $\int \cos^3 7x \sin^2 7x \, dx$

$$\int \cos^2 7x \sin^2 7x \cos 7x \, dx$$

$$= \int (1-\sin^2 7x) \sin^2 7x \cos 7x \, dx$$

$$= \frac{1}{7} \int (\sin^2 7x - \sin^4 7x)(7\cos 7x) \, dx$$

$(u = \sin 7x, \; du = 7\cos 7x \, dx)$

$$= \frac{1}{7}\left(\frac{1}{3}\sin^3 7x - \frac{1}{5}\sin^5 7x\right) + C$$

$$= \frac{1}{21}\sin^3 7x - \frac{1}{35}\sin^5 7x + C$$

(f) (3 marks) $\int_{\pi/3}^{\pi/2} \sin 5\theta \cos 3\theta \, d\theta$

$$= \int_{\pi/3}^{\pi/2} \frac{1}{2}(\sin 8\theta + \sin 2\theta) \, d\theta$$

Using product-to-sum formula (or could integrate by parts)

$$= \left[-\frac{1}{16}\cos 8\theta - \frac{1}{4}\cos 2\theta\right]_{\pi/3}^{\pi/2}$$

$$= \left(-\frac{1}{16} + \frac{1}{4}\right) - \left(\frac{1}{32} + \frac{1}{8}\right)$$

$$= \frac{1}{32}$$