CAMOSUN

Name: SOLUTIONS

Mark: ___/25

MATH 100
Assignment 4

1. Evaluate the limits. If they do not exist, then determine whether they are ∞ or $-\infty$ or neither.

(a) (2 marks) $\lim_{x\to-\infty} \dfrac{\sqrt{x^2+2}}{3x-9}$ $= \lim_{x\to-\infty} \dfrac{\sqrt{x^2}\sqrt{1+\frac{2}{x^2}}}{3x-9} = \lim_{x\to-\infty} \dfrac{-x\sqrt{1+\frac{2}{x^2}}}{3x-9}$

$= \lim_{x\to-\infty} \dfrac{-\sqrt{1+\frac{2}{x^2}}}{3-\frac{9}{x}} = -\frac{1}{3}$

(b) (1 mark) $\lim_{x\to\infty} \dfrac{2\cos(5x+3)}{x}$ $= 0$ by Squeeze Theorem

since $-1 \le \cos(5x+3) \le 1$ and so $-\frac{2}{x} \le \frac{2\cos(5x+3)}{x} \le \frac{2}{x}$

for $x>0$ and $\lim_{x\to\infty} -\frac{2}{x} = 0 = \lim_{x\to\infty} \frac{2}{x}$.

2. (3 marks) Find the equations of any horizontal, slant or vertical asymptotes of the graph of

$$f(x) = \frac{2x^4 - 8x^3 + 3x - 12}{x^3 - 2x^2 - 8x}.$$

$$f(x) = \frac{2x^3(x-4)+3(x-4)}{x(x^2-2x-8)} = \frac{(2x^3+3)(x-4)}{x(x+2)(x-4)} = \frac{2x^3+3}{x(x+2)} = \frac{2x^3+3}{x^2+2x}$$

$x = 0$ and $x = -2$ are vertical asymptotes (V.A.)
(i.e. nonremovable discontinuities)

$x = 4$ is a hole (i.e. removable discontinuity)

$$\begin{array}{r} 2x-4 \\ x^2+2x \overline{)\,2x^3 \qquad +3} \\ \underline{2x^3+4x^2} \\ -4x^2 \\ \underline{-4x^2-8x} \\ 8x+3 \end{array}$$

$$f(x) = 2x-4 + \frac{8x+3}{x^2+2x}$$

$\therefore$ $y = 2x-4$ is a slant asymptote

(there are no horizontal asymptotes)

3. (6 marks) Let

$$f(x) = \frac{x^2 + x - 2}{x^2}.$$

(a) Fully analyze f by finding its domain, intercepts, asymptotes, relative extrema and points of inflection, by testing it for symmetry, and by finding the intervals on which it is increasing or decreasing and where its graph is concave upward or concave downward.

1. domain is $\{x \mid x \neq 0\}$

2. $f(0)$ is undefined, $\therefore$ no y-int

3. $f(x) = \frac{(x+2)(x-1)}{x^2} = 0 \Rightarrow x = -2$ or $x = 1$ $\therefore (-2,0)$ and $(1,0)$ are x-int

4. $f(-x) = \frac{(-x)^2 + (-x) - 2}{(-x)^2} = \frac{x^2 - x - 2}{x^2} \neq \pm f(x)$ $\therefore$ no symmetry

5. not periodic

6. $x = 0$ is a V.A.

7. $\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \left(1 + \frac{1}{x} - \frac{2}{x^2}\right) = 1$ $\therefore y = 1$ is H.A.

8. $f'(x) = \frac{d}{dx}\left[1 + \frac{1}{x} - \frac{2}{x^2}\right] = -\frac{1}{x^2} + \frac{4}{x^3} = \frac{-(x-4)}{x^3} = 0$ at $x = 4$
(and undefined at $x = 0$)

$f(4) = \frac{9}{8}$ $\therefore (4, \frac{9}{8})$ is a critical point

9.

Interval	$(-\infty, 0)$	$(0, 4)$	$(4, \infty)$
f'	$-$	$+$	$-$
f	$\searrow$	$\nearrow$	$\searrow$

$\therefore (4, 9/8)$ is a local max by FDT

10. $f''(x) = \frac{d}{dx}\left[-\frac{1}{x^2} + \frac{4}{x^3}\right] = \frac{2}{x^3} - \frac{12}{x^4} = \frac{2(x-6)}{x^4} = 0$ at $x = 6$
(and undefined at $x = 0$)

$f(6) = \frac{10}{9}$ $\therefore (6, \frac{10}{9})$ is a possible P.O.I.

11.

Interval	$(-\infty, 0)$	$(0, 6)$	$(6, \infty)$
f''	$-$	$-$	$+$
f	$\cap$	$\cap$	$\cup$

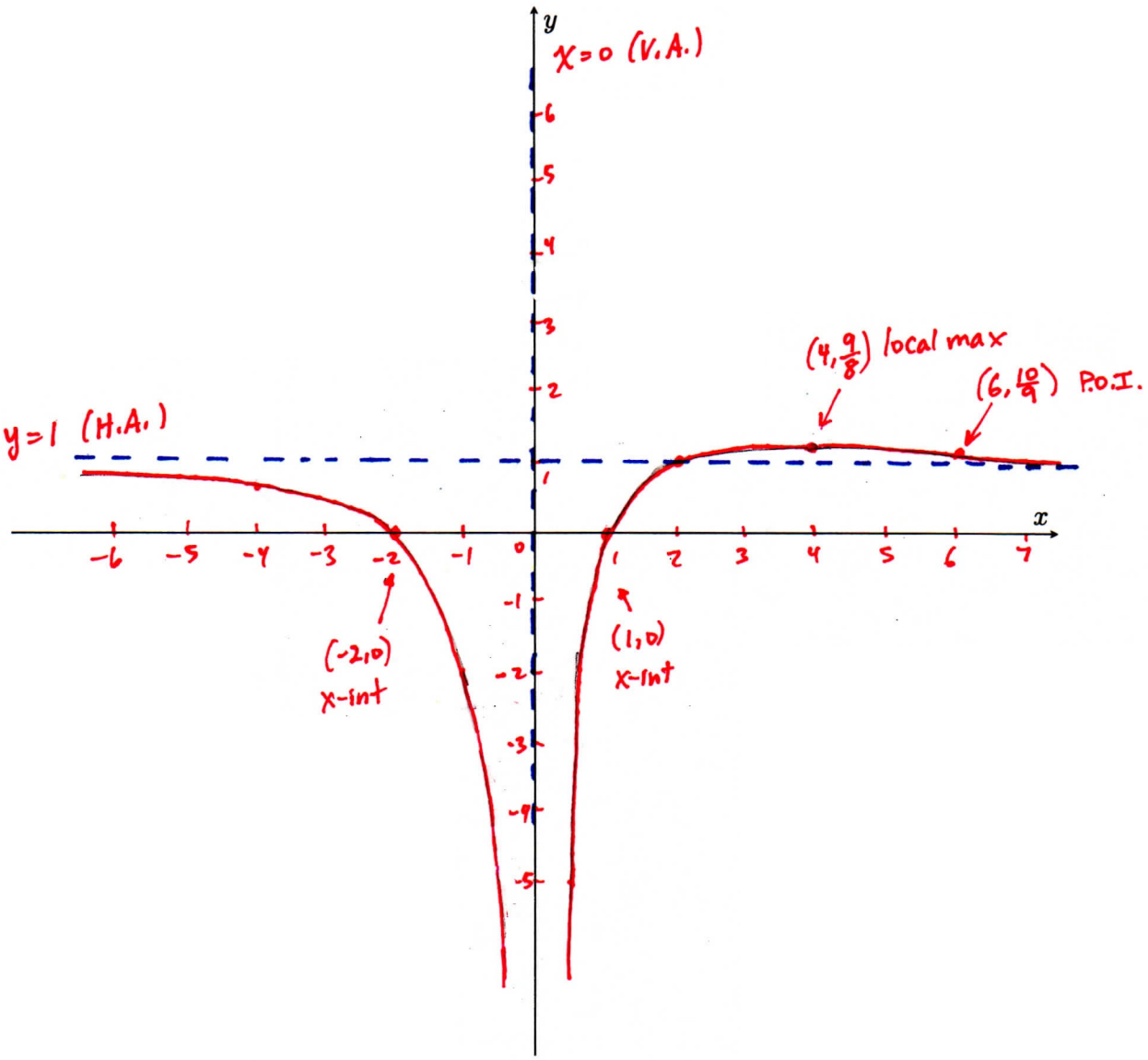
$\therefore (6, \frac{10}{9})$ is a P.O.I.

$f''(4) = -\frac{1}{64} < 0$ $\cap$ proving (again) by SDT that $(4, 9/8)$ is a local max.

12. Summary from charts in #9 and #11:

Interval	$(-\infty, 0)$	$(0, 4)$	$(4, 6)$	$(6, \infty)$
f	$\frown$	$\frown$	$\frown$	$\smile$

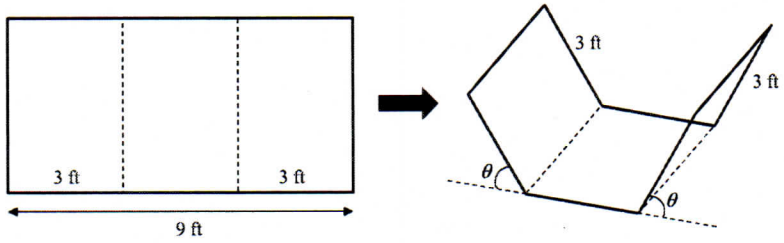
(b) Using your analysis from part (a), sketch the graph of f. Label any intercepts, relative extrema, points of inflection and asymptotes on your graph.



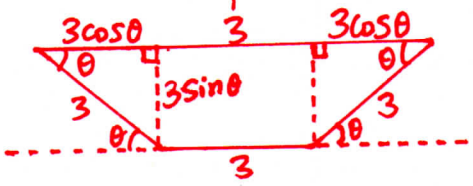
Extra points :

x	y
-4	0.625
-1	-2
0.5	-5
2	1

4. (4 marks) A 9 ft wide sheet of metal is used to make a water trough by bending up, at an angle θ, a 3 ft wide strip on each side. What angle θ would maximize the cross-sectional area, and therefore the volume, of the trough?



Cross-section is a trapezoid:



Area $A = \frac{1}{2}(b_1+b_2)h = \frac{1}{2}\left[(3+6\cos\theta)+3\right](3\sin\theta)$

$= \frac{3}{2}(6+6\cos\theta)\sin\theta = 9(1+\cos\theta)\sin\theta$ for $0 \le \theta \le \frac{\pi}{2}$

$A' = 9\left[(1+\cos\theta)\cos\theta + \sin\theta(-\sin\theta)\right] = 9(\cos\theta + \cos^2\theta - \sin^2\theta)$

$= 9(2\cos^2\theta + \cos\theta - 1) = 9(2\cos\theta - 1)(\cos\theta + 1)$

$\therefore A' = 0 \Rightarrow \cos\theta = \frac{1}{2}$ or $\cos\theta = -1$

$\therefore \theta = \frac{\pi}{3}$ is the only critical value in $[0, \frac{\pi}{2}]$.

Check endpoints and critical point (or use FDT or SDT):

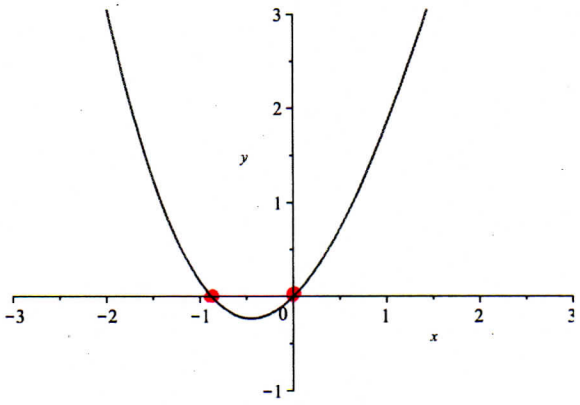
$A(0) = 0$

$A(\frac{\pi}{3}) = \frac{27\sqrt{3}}{4} \approx 11.69 \leftarrow$ MAX

$A(\frac{\pi}{2}) = 9$

$\therefore$ An angle of $\frac{\pi}{3}$ will maximize the cross-sectional area.

5. (3 marks) Find the zeros of $f(x) = x^2 + \sin x$. Use Newton's Method to approximate to at least 8 decimal places any zeros that cannot be found analytically.



Clearly $f(0) = 0$ and so $x = 0$ is a zero of f.

Let us use Newton's Method with an initial estimate of $x_1 = -1$ to approximate the other zero.

Since $f'(x) = 2x + \cos x$, then $x_{n+1} = x_n - \dfrac{x_n^2 + \sin x_n}{2x_n + \cos x_n}$.

$x_1 = -1$
$x_2 = -0.891\,395\,995$
$x_3 = -0.876\,984\,844$
$x_4 = -0.876\,726\,298$
$x_5 = -0.876\,726\,215$
$x_6 =$ same to 9 decimals

$\therefore$ 0 and $-0.876\,726\,215$ are zeros of f.

6. (3 marks) Use differentials or a linear approximation to approximate $7.98^{2/3}$. Express your answer as a fraction reduced to lowest terms.

Let $f(x) = x^{2/3}$. Then $f'(x) = \frac{2}{3x^{1/3}}$.

$f(x) \approx f(c) + f'(c)(x-c)$, $c=8$

$x^{2/3} \approx 8^{2/3} + \frac{2}{3(8^{1/3})}(x-8)$

$= 4 + \frac{1}{3}(x-8)$ for $x \approx 8$

$\therefore 7.98^{2/3} \approx 4 + \frac{1}{3}(7.98-8)$

$= 4 + \frac{1}{3}(-0.02)$

$= 4 - \frac{1}{3}\left(\frac{1}{50}\right) = \frac{599}{150}$

OR

$f(x+\Delta x) \approx f(x) + f'(x)\,dx$

$\Delta x = dx = 7.98 - 8 = -0.02$

$(x+\Delta x)^{2/3} \approx x^{2/3} + \frac{2}{3x^{1/3}}\,dx$ for small Δx

$7.98^{2/3} \approx 8^{2/3} + \frac{2}{3(8^{1/3})}(-0.02)$

$= 4 + \frac{1}{3}(-0.02)$

$= 4 - \frac{1}{3}\left(\frac{1}{50}\right) = \frac{599}{150}$

(Note: On calc., $\frac{599}{150} \approx 3.993333333$ and $7.98^{2/3} \approx 3.993330552$)

7. (3 marks) The area of an equilateral triangle having sides of length x is $A = \frac{\sqrt{3}}{4}x^2$.

(a) Express the area A of the triangle as a function of its perimeter P.

(b) Suppose the perimeter of the triangle is measured to be 50 cm with a possible error of 0.75 cm. Then using differentials, approximate the possible *propagated* and *percent* errors in computing the area of the triangle.

a) $P = 3x \Rightarrow x = \frac{P}{3}$ $\quad \therefore A = \frac{\sqrt{3}}{4}\left(\frac{P}{3}\right)^2 = \frac{\sqrt{3}}{36}P^2$

b) $P = 50\text{cm}$, $dP = \pm 0.75\text{cm}$ $\quad A = \frac{\sqrt{3}}{36}(50\text{cm})^2 \approx 120.28\text{cm}^2$

Propagated error: $\Delta A \approx dA = \frac{\sqrt{3}}{18}P\,dP = \frac{\sqrt{3}}{18}(50\text{cm})(\pm 0.75\text{cm})$

$\approx \pm 3.61\text{cm}^2$

Percent error: $\frac{\Delta A}{A} \approx \frac{dA}{A} = \frac{3.61\text{cm}^2}{120.28\text{cm}^2} = 0.03 = 3\%$

(or $\frac{\Delta A}{A} \approx \frac{dA}{A} = \frac{\frac{\sqrt{3}}{18}P\,dP}{\frac{\sqrt{3}}{36}P^2} = \frac{2\,dP}{P} = \frac{2(0.75\text{cm})}{50\text{cm}} = 0.03 = 3\%$)