CAMOSUN

Name: SOLUTIONS

Mark: ___/25

MATH 100
Assignment 3

1. (2 marks) Find $\frac{dy}{dx}$ by implicit differentiation.

$$2x\sin(xy^3) = x^4 + 2y - 7$$

$2x\cos(xy^3)\left(x\cdot 3y^2\frac{dy}{dx} + y^3\right) + 2\sin(xy^3) = 4x^3 + 2\frac{dy}{dx}$

$6x^2y^2\cos(xy^3)\frac{dy}{dx} + 2xy^3\cos(xy^3) + 2\sin(xy^3) = 4x^3 + 2\frac{dy}{dx}$

$\left(6x^2y^2\cos(xy^3) - 2\right)\frac{dy}{dx} = 4x^3 - 2xy^3\cos(xy^3) - 2\sin(xy^3)$

$$\frac{dy}{dx} = \frac{4x^3 - 2xy^3\cos(xy^3) - 2\sin(xy^3)}{6x^2y^2\cos(xy^3) - 2}$$

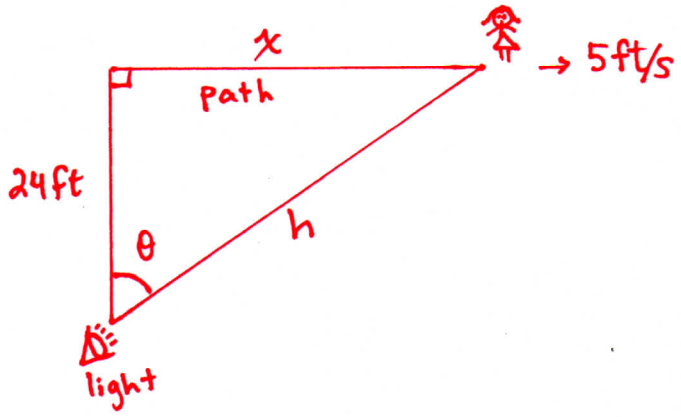
$$= \frac{2x^3 - xy^3\cos(xy^3) - \sin(xy^3)}{3x^2y^2\cos(xy^3) - 1}$$

2. (4 marks) A woman walks along a straight path at a speed of 5 ft/s. A searchlight is located on the ground 24 ft from the path and is kept focused on her. At what rate is the searchlight rotating when the woman is 16 ft from the point on the path closest to the searchlight?

Given: $\frac{dx}{dt} = 5\text{ft/s}$ (or -5ft/s)

Find: $\frac{d\theta}{dt}$ when $x = 16\text{ft}$

Equation: $\tan\theta = \frac{x}{24}$



$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{24}\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{24}\cos^2\theta \frac{dx}{dt}$$

When $x = 16$, $h = \sqrt{16^2+24^2} = \sqrt{832} = 8\sqrt{13}$

and so $\cos\theta = \frac{24}{h} = \frac{24}{8\sqrt{13}} = \frac{3}{\sqrt{13}}$

$$\therefore \frac{d\theta}{dt} = \frac{1}{24}\left(\frac{3}{\sqrt{13}}\right)^2(5) = \frac{15}{104} \approx 0.1442$$

$\therefore$ Angle is rotating at a rate of $\frac{15}{104}$ rad/s (approx. 8.26°/s)

(Note: The angle θ could be increasing or decreasing at that rate depending on whether the woman is walking away from or towards the search light.)

3. Find the absolute extrema of f on the indicated interval.

(a) (3 marks) $f(x) = -3x^{1/3}(x - 32)$ on $[-1, 27]$

f is continuous on $[-1, 27]$ $\quad f(x) = -3x^{4/3} + 96x^{1/3}$

$f'(x) = -4x^{1/3} + 32x^{-2/3} = -4x^{-2/3}(x-8) = \frac{-4(x-8)}{x^{2/3}}$

$f'(x) = 0$ at $x = 8$
$f'(x)$ undefined at $x = 0$
} both critical values in the interval $(-1, 27)$

$f(-1) = -99 \leftarrow$ MIN
$f(27) = 45$
$f(8) = 144 \leftarrow$ MAX
$f(0) = 0$

The max value of f is 144 (at $x = 8$)
and the min value of f is -99 (at $x = -1$).

(b) (3 marks) $f(x) = \cos 2x - 2\cos x$ on $[0, 2\pi]$

f is continuous on $[0, 2\pi]$

$f'(x) = -2\sin 2x + 2\sin x$

$= -4\sin x \cos x + 2\sin x$

$= -2\sin x (2\cos x - 1)$

$f'(x) = 0 \Rightarrow \sin x = 0$ or $\cos x = \frac{1}{2}$

$x = 0, \pi, 2\pi$ $x = \frac{\pi}{3}, \frac{5\pi}{3}$

$f(0) = -1$

$f(\pi) = 3$ ← MAX

$f(2\pi) = -1$

$f(\frac{\pi}{3}) = -\frac{3}{2}$ ← MIN

$f(\frac{5\pi}{3}) = -\frac{3}{2}$ ← MIN

The max value of f is 3 (at $x = \pi$)

and the min value of f is $-\frac{3}{2}$ (at $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$)

4. Determine whether the Mean Value Theorem can be applied to f on the indicated interval. If the Mean Value Theorem can be applied, then find all values of c that are guaranteed to exist. If it cannot be applied, then explain why not.

(a) (1 mark) $f(x) = x^{1/3} - 2x$ on $[-1, 8]$

f is continuous on $[-1, 8]$ ✓

$f'(x) = \frac{1}{3}x^{-2/3} - 2$

f is not differentiable at $x = 0$, which is in $(-1, 8)$

$\therefore$ MVT cannot be applied

(b) (3 marks) $f(x) = 2\sqrt{5 - x} + 3$ on $[1, 5]$

f is continuous on $[1, 5]$ ✓

$f'(x) = \frac{-1}{\sqrt{5-x}}$

f is differentiable on $(1, 5)$ ✓

(Note: f is not differentiable at the endpoint $x = 5$, but that is ok.)

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{f(5) - f(1)}{5 - 1} = \frac{3 - 7}{4} = \frac{-4}{4} = -1$$

Solving $f'(x) = -1$:

$$\frac{-1}{\sqrt{5-x}} = -1$$

$$\sqrt{5-x} = 1$$

$$5 - x = 1$$

$$x = 4$$

$\therefore c = 4$ (which is in $(1, 5)$)

5. (3 marks) Let

$$f(x) = \frac{1}{2}x + \cos x$$

for x in the interval $(0, 2\pi)$. Find the open intervals on which f is increasing or decreasing and find the coordinates of all critical points. Using the First Derivative Test, classify each critical point as a relative maximum, relative minimum or neither.

$f'(x) = \frac{1}{2} - \sin x = 0 \implies \sin x = \frac{1}{2} \implies x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

$f(\pi/6) = \frac{\pi}{12} + \frac{\sqrt{3}}{2} \approx 1.13$ and $f(\frac{5\pi}{6}) = \frac{5\pi}{12} - \frac{\sqrt{3}}{2} \approx 0.44$

$\therefore \left(\frac{\pi}{6}, \frac{\pi}{12} + \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{5\pi}{6}, \frac{5\pi}{12} - \frac{\sqrt{3}}{2}\right)$ are critical points

Interval	$(0, \frac{\pi}{6})$	$(\frac{\pi}{6}, \frac{5\pi}{6})$	$(\frac{5\pi}{6}, 2\pi)$
Sign of f'	+	−	+
f	↗	→	↗

f is increasing on $(0, \pi/6)$ and $(\frac{5\pi}{6}, 2\pi)$
and decreasing on $(\frac{\pi}{6}, \frac{5\pi}{6})$

By FDT, f has a relative max at $\left(\frac{\pi}{6}, \frac{\pi}{12} + \frac{\sqrt{3}}{2}\right)$
and a relative min at $\left(\frac{5\pi}{6}, \frac{5\pi}{12} - \frac{\sqrt{3}}{2}\right)$

6. (3 marks) Using the Second Derivative Test, where applicable, find all the relative extrema of

$$f(x) = 12x^5 - 45x^4 + 80x^3 - 360x^2 + 100.$$

$f'(x) = 60x^4 - 180x^3 + 240x^2 - 720x$

$= 60x(x^3 - 3x^2 + 4x - 12)$

$= 60x[x^2(x-3) + 4(x-3)]$

$= 60x(x^2+4)(x-3)$

$f'(x) = 0$ at $x = 0$ or $x = 3$ (critical values)

$f(0) = 100$ and $f(3) = -1709$

$\therefore$ $(0, 100)$ and $(3, -1709)$ are critical points

$f''(x) = 240x^3 - 540x^2 + 480x - 720$

$f''(0) = -720 < 0$ $\cap$ $\therefore$ $(0,100)$ is a relative max

$f''(3) = 2340 > 0$ $\cup$ $\therefore$ $(3,-1709)$ is a relative min by SDT

7. (3 marks) Find the open intervals on which the graph of

$$f(x) = \frac{4x}{x^2+12}$$

is concave upward or concave downward and find the coordinates of all points of inflection.

$$f'(x) = \frac{(x^2+12)(4) - 4x(2x)}{(x^2+12)^2} = \frac{4x^2+48-8x^2}{(x^2+12)^2} = \frac{-4x^2+48}{(x^2+12)^2}$$

$$f''(x) = \frac{(x^2+12)^2(-8x) - (-4x^2+48)(2)(x^2+12)(2x)}{(x^2+12)^4} = \frac{-4x(x^2+12)\left[2(x^2+12)+(-4x^2+48)\right]}{(x^2+12)^4}$$

$$= \frac{-4x(2x^2+24-4x^2+48)}{(x^2+12)^3} = \frac{-4x(-2x^2+72)}{(x^2+12)^3} = \frac{8x(x^2-36)}{(x^2+12)^3} = \frac{8x(x-6)(x+6)}{(x^2+12)^3}$$

$f''(x)$ is defined for all x and $f''(0) = 0$ at $x = 0, 6, -6$, where $f(0) = 0$, $f(6) = \frac{1}{2}$ and $f(-6) = -\frac{1}{2}$

Interval	$(-\infty, -6)$	$(-6, 0)$	$(0, 6)$	$(6, \infty)$
Sign of f''	$-$	$+$	$-$	$+$
f	$\cap$	$\cup$	$\cap$	$\cup$

Graph is concave up on $(-6, 0)$ and $(6, \infty)$ and concave down on $(-\infty, -6)$ and $(0, 6)$.

$(0, 0)$, $(6, \frac{1}{2})$ and $(-6, -\frac{1}{2})$ are points of inflection