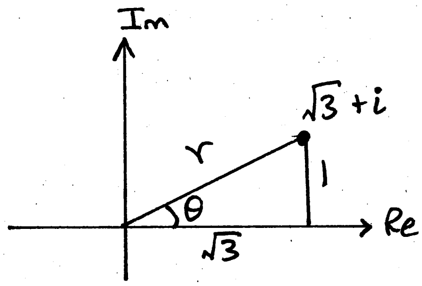
CAMOSUN

Name: SOLUTIONS

MATH 251 (Winter, 2019)
Term Test 3
by George Ballinger

Answer the questions in the space provided.
This test has 5 questions for a total of 25 marks.

1. (4 marks) Evaluate $(\sqrt{3}+i)^{11}$. Express your answer in both polar and rectangular form, using exact values in both cases.

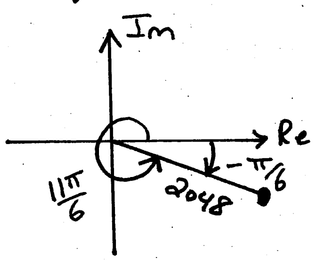


$r = \sqrt{(\sqrt{3})^2+1^2} = 2$ and $\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$ or $30°$

$\therefore \sqrt{3}+i = 2e^{i\pi/6}$

$(\sqrt{3}+i)^{11} = \left(2e^{i\pi/6}\right)^{11} = 2^{11}e^{i\frac{11\pi}{6}} = \underline{2048e^{i\frac{11\pi}{6}}}$ or $2048\angle\frac{11\pi}{6}$

or $2048\angle 330°$



(instead of $\frac{11\pi}{6}$ or $330°$ we could use, for example, $-\frac{\pi}{6}$ or $-30°$)

In rectangular form

$$2048e^{i\frac{11\pi}{6}} = 2048\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$$
$$= 2048\left(\frac{\sqrt{3}}{2} + i\left(\frac{-1}{2}\right)\right)$$
$$= \underline{1024\sqrt{3} - 1024i}$$

2. (4 marks) Use Cramer's Rule to find the y-value in the solution of the following system of linear equations. You do not need to solve for the other variables.

$$\begin{cases} 2x \qquad - 6z = 10 \\ x + 4y - 2z = -8 \\ 5x + 2y + \ z = 3 \end{cases}$$

$$\underbrace{\begin{bmatrix} 2 & 0 & -6 \\ 1 & 4 & -2 \\ 5 & 2 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 10 \\ -8 \\ 3 \end{bmatrix}}_{\vec{b}}$$

$$\det(A) = \begin{vmatrix} 2 & 0 & -6 \\ 1 & 4 & -2 \\ 5 & 2 & 1 \end{vmatrix} = 2\begin{vmatrix} 4 & -2 \\ 2 & 1 \end{vmatrix} + (-6)\begin{vmatrix} 1 & 4 \\ 5 & 2 \end{vmatrix} = 2(8) - 6(-18) = 124$$

$$\det(A_2(\vec{b})) = \begin{vmatrix} 2 & 10 & -6 \\ 1 & -8 & -2 \\ 5 & 3 & 1 \end{vmatrix} = 2\begin{vmatrix} -8 & -2 \\ 3 & 1 \end{vmatrix} - 10\begin{vmatrix} 1 & -2 \\ 5 & 1 \end{vmatrix} + (-6)\begin{vmatrix} 1 & -8 \\ 5 & 3 \end{vmatrix}$$

$$= 2(-2) - 10(11) - 6(43) = -372$$

$$\therefore \ y = \frac{\det(A_2(\vec{b}))}{\det(A)} = \frac{-372}{124} = -3$$

3. (5 marks) Let $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$.

(a) Find the eigenvalues and corresponding eigenspaces of A.

(b) If possible, find a diagonal matrix D and an invertible matrix P so that $P^{-1}AP = D$. If it's not possible, then explain why.

a) $\det(A-\lambda I) = \begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = (4-\lambda)(2-\lambda)-3 = 8-6\lambda+\lambda^2-3 = \lambda^2-6\lambda+5$

$= (\lambda-1)(\lambda-5)=0 \implies \lambda_1=1$ or $\lambda_2=5$ (eigenvalues)

$\underline{\lambda_1=1}$ $A-\lambda_1 I = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$ Solve $(A-\lambda_1 I)\vec{x}=\vec{0}$

$\left[\begin{array}{cc|c} 3 & 1 & 0 \\ 3 & 1 & 0 \end{array}\right] \xrightarrow[R_2-R_1]{\frac{1}{3}R_1} \left[\begin{array}{cc|c} 1 & 1/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $y=t,\ x=-\frac{1}{3}t$

$\therefore \vec{x} = \begin{bmatrix} -1/3\,t \\ t \end{bmatrix} = \frac{1}{3}t\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ $\therefore$ an eigenvector is $\vec{x}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$E_{\lambda_1} = \text{span}\left(\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right)$

$\underline{\lambda_2=5}$ $A-\lambda_2 I = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}$ Solve $(A-\lambda_2 I)\vec{x}=\vec{0}$

$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 3 & -3 & 0 \end{array}\right] \xrightarrow[R_2+3R_1]{-R_1} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $y=t,\ x=t$

$\therefore \vec{x} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\therefore$ an eigenvector is $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$E_{\lambda_2} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

b) Let $D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ and $P = \begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix}$.

4. (5 marks) Consider the following matrix.

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & x \end{bmatrix}$$

For what value(s) of x (if any) is A *not* diagonalizable? Justify.

Since A is uppertriangular, its eigenvalues are the diagonal entries $2, 1,$ and x. If $x \neq 1$ and $x \neq 2$, then A has 3 distinct eigenvalues and $\therefore$ 3 linearly independent eigenvectors and will be diagonalizable.

If $x=1$, then $\lambda=1$ is an eigenvalue of A with algebraic multiplicity 2. A will be diagonalizable iff the geometric multiplicity of $\lambda=1$ is also 2. The same reasoning applies if $x=2$.

Check $x=1$. $A=\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\lambda=1$; $A-\lambda I = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Solve $(A-\lambda I)\vec{x} = \vec{0}$ $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ (already RREF)

$y=s$, $z=t$, $x=-t$ $\therefore \vec{x} = \begin{bmatrix} -t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$E_\lambda = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$. Since geo. mult. of $\lambda=1$ is 2, matrix is diagonalizable

Check $x=2$. $A=\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $\lambda=2$, $A-\lambda I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Solve $(A-\lambda I)\vec{x} = \vec{0}$ $\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x=t$, $y=0$, $z=0$ $\therefore \vec{x} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$E_\lambda = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$. Since geo. mult. of $\lambda=2$ is only 1 then matrix is NOT diagonalizable

$\therefore \boxed{x=2}$

5. (7 marks) Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 5 \\ -5 \\ 9 \end{bmatrix}$$

and define $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $W = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$.

(a) Show that $\mathcal{B}$ is an orthogonal basis for $\mathbb{R}^3$.

(b) Express $\mathbf{v}$ as a linear combination of the vectors in $\mathcal{B}$. In other words, find $[\mathbf{v}]_{\mathcal{B}}$.

(c) Find a basis for $W^{\perp}$.

(d) Find $\text{proj}_W(\mathbf{v})$ and $\text{perp}_W(\mathbf{v})$.

(e) Construct an orthonormal basis for $\mathbb{R}^3$ using the vectors from $\mathcal{B}$.

Note: If you need more space, you may continue your answer on the back of this page.

a) $\left.\begin{aligned} \vec{V}_1 \cdot \vec{V}_2 &= 3-4+1=0 \\ \vec{V}_1 \cdot \vec{V}_3 &= 2+2-4=0 \\ \vec{V}_2 \cdot \vec{V}_3 &= 6-2-4=0 \end{aligned}\right\}$ $\therefore$ B is an orthogonal set. Vectors in $\mathcal{B}$ are linearly independent, so $\mathcal{B}$ is an orthogonal basis for $\mathbb{R}^3$.

b) $$\vec{V} = \underbrace{\left(\frac{\vec{V}\cdot\vec{V}_1}{\vec{V}_1\cdot\vec{V}_1}\right)\vec{V}_1}_{\text{proj}_{\vec{V}_1}(\vec{V})} + \underbrace{\left(\frac{\vec{V}\cdot\vec{V}_2}{\vec{V}_2\cdot\vec{V}_2}\right)\vec{V}_2}_{\text{proj}_{\vec{V}_2}(\vec{V})} + \underbrace{\left(\frac{\vec{V}\cdot\vec{V}_3}{\vec{V}_3\cdot\vec{V}_3}\right)\vec{V}_3}_{\text{proj}_{\vec{V}_3}(\vec{V})} = \left(\frac{24}{6}\right)\vec{V}_1 + \left(\frac{14}{14}\right)\vec{V}_2 + \left(\frac{-21}{21}\right)\vec{V}_3$$

$$= 4\vec{V}_1 + \vec{V}_2 - \vec{V}_3 \qquad \therefore [\vec{V}]_B = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$$

c) A basis for $W^{\perp}$ is $\{\vec{V}_3\}$.

d) $$\text{proj}_W(\vec{V}) = \text{proj}_{\vec{V}_1}(\vec{V}) + \text{proj}_{\vec{V}_2}(\vec{V}) = 4\vec{V}_1 + \vec{V}_2 = 4\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \\ 5 \end{bmatrix}$$

$$\text{perp}_W(\vec{V}) = \underbrace{\vec{V} - \text{proj}_W(\vec{V})}_{\text{or } \text{proj}_{W^{\perp}}(\vec{V})} = -\vec{V}_3 = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$

e) $\|\vec{V}_1\| = \sqrt{6}$, $\|\vec{V}_2\| = \sqrt{14}$, $\|\vec{V}_3\| = \sqrt{21}$

$\therefore$ an orthonormal basis for $\mathbb{R}^3$ is $\left\{ \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{14}}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{21}}\begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix} \right\}$.