Name: SOLUTIONS

Mark: $\overline{\phantom{00}}$ **25**

# MATH 251 (Winter, 2022)

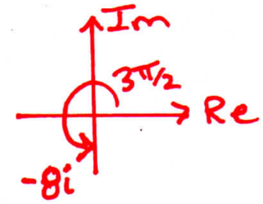
## Test 3

1. (5 marks) Find all complex number solutions of the equation

$$z^3 + 8i = 0.$$

Express your answer(s) in the rectangular form $a + bi$ using exact values.

$z^3 = -8i = 8e^{\frac{3\pi}{2}i}$




Need to find cube roots of $-8i$.

$1^{st}$ root is $z_1 = 8^{\frac{1}{3}} e^{\frac{\pi}{2}i} = 2e^{\frac{\pi}{2}i} = 2(0+i) = 2i$

$2^{nd}$ root is $z_2 = 8^{\frac{1}{3}} e^{i\left(\frac{\pi}{2}+\frac{2\pi}{3}\right)} = 2e^{\frac{7\pi}{6}i} = 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$

$= 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = -\sqrt{3} - i$

$3^{rd}$ root is $z_3 = 8^{\frac{1}{3}} e^{i\left(\frac{\pi}{2}+\frac{4\pi}{3}\right)} = 2e^{\frac{11\pi}{6}i} = 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$

$= 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \sqrt{3} - i$

$\therefore\ z = 2i,\ -\sqrt{3} - i,\ \sqrt{3} - i$

2. (6 marks) Let $A = \begin{bmatrix} 3 & -2 & -8 \\ 2 & 6 & 1 \\ -1 & 0 & 2 \end{bmatrix}$.

(a) Calculate $\text{tr}(A)$, the trace of matrix $A$.

(b) Use the cofactor method to find $A^{-1}$.

(c) Find the volume of the parallelepiped determined by the column vectors of $A$.

a) $tr(A) = 3+6+2 = 11$

b)

$$C = \begin{bmatrix} \begin{vmatrix} 6 & 1 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 6 \\ -1 & 0 \end{vmatrix} \\ -\begin{vmatrix} -2 & -8 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 3 & -8 \\ -1 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & -2 \\ -1 & 0 \end{vmatrix} \\ \begin{vmatrix} -2 & -8 \\ 6 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & -8 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 3 & -2 \\ 2 & 6 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 12 & -5 & 6 \\ 4 & -2 & 2 \\ 46 & -19 & 22 \end{bmatrix}$$

$$C^T = \begin{bmatrix} 12 & 4 & 46 \\ -5 & -2 & -19 \\ 6 & 2 & 22 \end{bmatrix}$$

$$\det(A) = (3)(12) + (-2)(-5) + (-8)(6) = -2$$

$$A^{-1} = \frac{1}{\det(A)} C^T = -\tfrac{1}{2} C^T = \begin{bmatrix} -6 & -2 & -23 \\ 5/2 & 1 & 19/2 \\ -3 & -1 & -11 \end{bmatrix}$$

c) Volume $= |\det(A)| = |-2| = 2$

3. (6 marks) Diagonalize the matrix

$$A = \begin{bmatrix} 1 & -4 \\ -1 & -2 \end{bmatrix}$$

by factoring it into a product $A = PDP^{-1}$, where $P$ is an invertible matrix and $D$ is a diagonal matrix.

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -4 \\ -1 & -2-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (1-\lambda)(-2-\lambda) - 4 = \lambda^2 + \lambda - 2 - 4 = \lambda^2 + \lambda - 6 = (\lambda-2)(\lambda+3)$$

$$\det(A-\lambda I) = 0 \rightarrow \lambda_1 = 2, \quad \lambda_2 = -3 \quad \text{(eigenvalues)}$$

$\underline{\lambda_1 = 2}$ $A - \lambda_1 I = \begin{bmatrix} -1 & -4 \\ -1 & -4 \end{bmatrix}$ Solve $(A-\lambda_1 I)\vec{x} = \vec{0}$

$$\left[\begin{array}{cc|c} -1 & -4 & 0 \\ -1 & -4 & 0 \end{array}\right] \xrightarrow[R_2 - R_1]{-R_1} \left[\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \left.\begin{array}{l} x = -4t \\ y = t \end{array}\right\} \vec{x} = t\begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

$\therefore$ eigenvector is $\vec{x}_1 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

$\underline{\lambda_2 = -3}$ $A - \lambda_2 I = \begin{bmatrix} 4 & -4 \\ -1 & 1 \end{bmatrix}$ Solve $(A-\lambda_2 I)\vec{x} = \vec{0}$

$$\left[\begin{array}{cc|c} 4 & -4 & 0 \\ -1 & 1 & 0 \end{array}\right] \xrightarrow[R_2 + \frac{1}{4}R_1]{\frac{1}{4}R_1} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \left.\begin{array}{l} x = t \\ y = t \end{array}\right\} \vec{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\therefore$ eigenvector is $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Let $D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ and $P = \begin{bmatrix} -4 & 1 \\ 1 & 1 \end{bmatrix}$.

Then $P^{-1} = \frac{1}{-5}\begin{bmatrix} 1 & -1 \\ -1 & -4 \end{bmatrix} = \begin{bmatrix} -1/5 & 1/5 \\ 1/5 & 4/5 \end{bmatrix}$ and so

$$A = \begin{bmatrix} -4 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} -1/5 & 1/5 \\ 1/5 & 4/5 \end{bmatrix}$$

4. (8 marks) Let

$$\mathbf{v}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}0\\0\\-1\\1\end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$$

and define $W = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$.

(a) Show that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for $W$.

(b) Find an orthonormal basis for $W$.

(c) Find $\text{proj}_W(\mathbf{v})$ and $\text{perp}_W(\mathbf{v})$.

a) $\vec{v}_1 \cdot \vec{v}_2 = 0$, $\therefore$ $\vec{v}_1$ and $\vec{v}_2$ are orthogonal (and L.I.) and form basis for W

b) $\|\vec{v}_1\| = \sqrt{4} = 2$ and $\|\vec{v}_2\| = \sqrt{2}$

$\therefore$ orthonormal basis is $\left\{\frac{1}{2}\vec{v}_1, \frac{1}{\sqrt{2}}\vec{v}_2\right\} = \left\{\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}, \begin{bmatrix}0\\0\\-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}\right\}$

c) $\text{proj}_W(\vec{v}) = \text{proj}_{\vec{v}_1}(\vec{v}) + \text{proj}_{\vec{v}_2}(\vec{v}) = \left(\frac{\vec{v}_1 \cdot \vec{v}}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 + \left(\frac{\vec{v}_2 \cdot \vec{v}}{\vec{v}_2 \cdot \vec{v}_2}\right)\vec{v}_2$

$= \frac{10}{4}\vec{v}_1 + \frac{1}{2}\vec{v}_2 = \frac{5}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0\\0\\-1\\1\end{bmatrix} = \begin{bmatrix}5/2\\5/2\\2\\3\end{bmatrix}$

and $\text{perp}_W(\vec{v}) = \vec{v} - \text{proj}_W(\vec{v}) = \begin{bmatrix}1\\2\\3\\4\end{bmatrix} - \begin{bmatrix}5/2\\5/2\\2\\3\end{bmatrix} = \begin{bmatrix}-3/2\\-1/2\\1\\1\end{bmatrix}$

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(d) Find a basis for $W^{\perp}$.

(e) Is your basis from part (d) orthogonal? Briefly explain.

d) $W^{\perp} = \text{null}\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\right)$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow{\begin{subarray}{l} R_1+R_2 \\ -R_2 \end{subarray}} \left[\begin{array}{cccc|c} \boxed{1} & 1 & 0 & 2 & 0 \\ 0 & 0 & \boxed{1} & -1 & 0 \end{array}\right] \qquad \left.\begin{array}{l} x_2 = s \\ x_4 = t \end{array}\right\} \text{free}$$

$$\left.\begin{array}{l} x_1 = -s-2t \\ x_3 = t \end{array}\right\} \text{leading} \qquad \therefore \vec{x} = s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Basis for $W^{\perp}$ is $\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right\}$

e) No, since $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 2 \neq 0$ (vectors not orthogonal)