CAMOSUN

Name: SOLUTIONS

MATH 251 (Winter, 2019)
Term Test 2
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Answer the questions in the space provided. This test has 6 questions for a total of 25 marks.

1. (3 marks) Find the matrix A that satisfies the following.

$$(I_2 + 3A^T)^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$I_2 + 3A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

$$3A^T = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 3/2 & -3/2 \end{bmatrix}$$

$$A^T = \frac{1}{3}\begin{bmatrix} -3 & 1 \\ 3/2 & -3/2 \end{bmatrix} = \begin{bmatrix} -1 & 1/3 \\ 1/2 & -1/2 \end{bmatrix}$$

$$\therefore\ A = \begin{bmatrix} -1 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{2} \end{bmatrix}$$

2. (3 marks) Solve the given matrix equation for X. Assume that all matrices are invertible. Simplify your answer.

$$BX^{-1} = (A - I)(A^{-1} + I) + A^{-1}$$

$$BX^{-1} = AA^{-1} + AI - IA^{-1} - II + A^{-1}$$
$$= I + A - A^{-1} - I + A^{-1} = A$$

$$B^{-1}BX^{-1} = B^{-1}A$$
$$X^{-1} = B^{-1}A$$
$$X = (B^{-1}A)^{-1} = A^{-1}B$$

3. (5 marks) Consider the following system of linear equations.

$$\begin{cases} x - \ \ y - 3z = -2 \\ x - 2y - 5z = 3 \\ -x + \ \ y + 4z = 10 \end{cases}$$

(a) Express the linear system in the matrix form: $A\mathbf{x} = \mathbf{b}$.

(b) Find A^{-1} using the Gauss-Jordan method for computing the inverse.

(c) Use A^{-1} to solve the system.

a) $$\begin{bmatrix} 1 & -1 & -3 \\ 1 & -2 & -5 \\ -1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 10 \end{bmatrix}$$

b) $$\left[\begin{array}{ccc|ccc} 1 & -1 & -3 & 1 & 0 & 0 \\ 1 & -2 & -5 & 0 & 1 & 0 \\ -1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{R_2 - R_1 \\ R_3 + R_1}} \left[\begin{array}{ccc|ccc} 1 & -1 & -3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{R_1 - R_2 \\ -R_2}} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{\substack{R_1 + R_3 \\ R_2 - 2R_3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & 1 \\ 0 & 1 & 0 & -1 & -1 & -2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

A^{-1}

$$\therefore A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & -1 & -2 \\ 1 & 0 & 1 \end{bmatrix}$$

c) $$\vec{x} = A^{-1}b = \begin{bmatrix} 3 & -1 & 1 \\ -1 & -1 & -2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ -21 \\ 8 \end{bmatrix}$$

i.e. $x = 1$
$y = -21$
$z = 8$

4. (4 marks) Find an LU-factorization of the following matrix.

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 8 & -6 & -4 \\ -4 & 3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 8 & -6 & -4 \\ -4 & 3 & 1 \end{bmatrix} \rightarrow \begin{matrix} R_2 - 8R_1 \\ R_3 + 4R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 5 \\ 0 & -22 & -44 \\ 0 & 11 & 21 \end{bmatrix} \rightarrow R_3 + \tfrac{1}{2}R_2 \begin{bmatrix} 1 & 2 & 5 \\ 0 & -22 & -44 \\ 0 & 0 & -1 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ -4 & -\tfrac{1}{2} & 1 \end{bmatrix} \quad \begin{matrix} R_3 + 8R_1 \\ R_3 - 4R_1 \quad R_3 - \tfrac{1}{2}R_2 \end{matrix}$$

$$\therefore A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ -4 & -\tfrac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ 0 & -22 & -44 \\ 0 & 0 & -1 \end{bmatrix}$$

5. (5 marks) Consider the following matrix A with its RREF.

$$A = \begin{bmatrix} 1 & 4 & -1 & 8 & 13 \\ -3 & 0 & -9 & -7 & -17 \\ 6 & 2 & 16 & 1 & 6 \\ 5 & -1 & 16 & 3 & 12 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find bases for the row space, column space and null space of A.

(b) Find rank(A) and nullity(A).

a) basis for row(A) is $\{[1\ 0\ 3\ 0\ 1], [0\ 1\ -1\ 0\ -1], [0\ 0\ 0\ 1\ 2]\}$

basis for col(A) is $\left\{\begin{bmatrix}1\\-3\\6\\5\end{bmatrix}, \begin{bmatrix}4\\0\\2\\-1\end{bmatrix}, \begin{bmatrix}8\\-7\\1\\3\end{bmatrix}\right\}$

basis for null(A):

$$\begin{cases} x_1 + 3x_3 + x_5 = 0 \\ x_2 - x_3 - x_5 = 0 \\ x_4 + 2x_5 = 0 \end{cases}$$

Free variables x_3, x_5

Let $x_3 = s$ and $x_5 = t$

then $x_1 = -3s - t$

$x_2 = s + t$

$x_4 = -2t$

$$\therefore \vec{x} = s\begin{bmatrix}-3\\1\\1\\0\\0\end{bmatrix} + t\begin{bmatrix}-1\\1\\0\\-2\\1\end{bmatrix}$$

$\therefore$ basis for null(A) is $\left\{\begin{bmatrix}-3\\1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\0\\-2\\1\end{bmatrix}\right\}$

b) rank(A) = 3

nullity(A) = 2

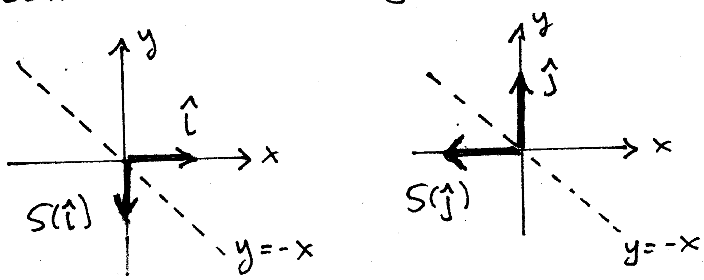
6. (5 marks) Consider the linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ that rotates a vector counterclockwise by $60°$ and then reflects it about the line $y = -x$.

   (a) Find the standard matrix of T.

   (b) Use your answer in part (a) to find $T(\mathbf{v})$, where $\mathbf{v} = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$. Give an exact answer.

a) Rotation $R_{60°}$: $A = \begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

Reflection S about $y=-x$:



$S(\hat{\imath}) = -\hat{\jmath} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ $\quad S(\hat{\jmath}) = -\hat{\imath} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $\quad \therefore B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

Transformation $T = S \circ R_{60°}$:

Standard matrix is

$$BA = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\sqrt{3}/2 & -1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$$

b) $$T\left(\begin{bmatrix} -4 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} -\sqrt{3}/2 & -1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 2\sqrt{3}-3 \\ 2+3\sqrt{3} \end{bmatrix} \approx \begin{bmatrix} 0.464 \\ 7.196 \end{bmatrix}$$