CAMOSUN

Name: SOLUTIONS

Mark: $\overline{}$ **25**

MATH 251 (Winter, 2022)

Test 2

1. (4 marks) Factor $A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix}$ into a product of elementary matrices.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - 4R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 & 1 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \qquad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -4 & 1 & 0 \end{array}\right] \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 1 & -4 & 1 & 0 \end{array}\right] \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A^{-1} = E_3 E_2 E_1$

$$\therefore A = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ (Answers vary)

Note: Since the question does not ask for A^{-1}, it's sufficient to RREF matrix A rather than the augmented matrix $[A|I]$.

2. (3 marks) Solve the system of equations

$$\left\{\begin{array}{rrrrl} 11x_1 + & 6x_2 - & 3x_3 - & 7x_4 = & 3 \\ -26x_1 - & 13x_2 + & 8x_3 + & 16x_4 = & -1 \\ -61x_1 - & 31x_2 + & 18x_3 + & 38x_4 = & 0 \\ -x_1 - & x_2 & + & x_4 = & 4 \end{array}\right\} A\vec{x} = \vec{b}$$

using the fact that

$$\underbrace{\begin{bmatrix} 11 & 6 & -3 & -7 \\ -26 & -13 & 8 & 16 \\ -61 & -31 & 18 & 38 \\ -1 & -1 & 0 & 1 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} 2 & 3 & -1 & 4 \\ 4 & -3 & 2 & 0 \\ 1 & 5 & -2 & 3 \\ 6 & 0 & 1 & 5 \end{bmatrix}}_{A^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{I_4}.$$

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 2 & 3 & -1 & 4 \\ 4 & -3 & 2 & 0 \\ 1 & 5 & -2 & 3 \\ 6 & 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 19 \\ 15 \\ 10 \\ 38 \end{bmatrix}$$

3. (8 marks) A matrix B is said to be a square root of a matrix A if $B^2 = A$. Consider a 2×2 matrix B having the form

$$B = \begin{bmatrix} 3 & k^2 \\ -k & -3 \end{bmatrix},$$

where k is a real number. Find the value(s) of k (if any) if

(a) B is a square root of the identity matrix I_2.

(b) $B\mathbf{x} = 6\mathbf{x}$, where $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(c) B is a symmetric matrix.

(d) B is an invertible matrix.

a) $B^2 = \begin{bmatrix} 3 & k^2 \\ -k & -3 \end{bmatrix}\begin{bmatrix} 3 & k^2 \\ -k & -3 \end{bmatrix} = \begin{bmatrix} 9-k^3 & 0 \\ 0 & 9-k^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow 9-k^3=1 \Rightarrow k^3=8 \quad \therefore \boxed{k=2}$

b) $B\vec{x} = \begin{bmatrix} 3 & k^2 \\ -k & -3 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 9+k^2 \\ -3k-3 \end{bmatrix} = \begin{bmatrix} 18 \\ 6 \end{bmatrix} = 6\begin{bmatrix} 3 \\ 1 \end{bmatrix} = 6\vec{x}$

$\Rightarrow 9+k^2=18$ and $-3k-3=6$

$k^2=9$ $\qquad -3k=9$

$k=\pm 3$ $\qquad k=-3$

$\therefore \boxed{k=-3}$

c) $B^T = B \Rightarrow k^2=-k \Rightarrow k^2+k=0 \Rightarrow k(k+1)=0$

$\therefore \boxed{k=0 \text{ or } k=-1}$

d) $\det(B) = -9+k^3 = 0 \Rightarrow k=\sqrt[3]{9}$. Need $\det(B) \neq 0$

$\therefore$ $\boxed{\text{K is any real number except } \sqrt[3]{9}}$

4. (6 marks) Suppose

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 2 & 1 & 3 & 2 & 5 \\ 3 & 2 & 5 & 4 & 7 \\ 4 & 2 & 6 & 1 & 7 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \\ 2 & 3 & 5 & 6 \\ 2 & 2 & 4 & 1 \\ 2 & 5 & 7 & 7 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for the column space of A **consisting of columns of** A.

(b) Find a basis for the row space of A **consisting of rows of** A.

(c) Find a basis for the null space of A.

(d) Find rank(A) and nullity(A).

a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 4 \\ 1 \end{bmatrix} \right\}$

b) $\{ [1\ 1\ 2\ 2\ 2], [2\ 1\ 3\ 2\ 5], [4\ 2\ 6\ 1\ 7] \}$

(using $\text{row}(A) = \text{col}(A^T)$)

c) Solve $A\vec{x} = \vec{0}$. From RREF $x_3 = s$ and $x_5 = t$ are free variables and $x_1 = -s - 3t$, $x_2 = -s + 3t$ and $x_4 = -t$.

$$\vec{x} = s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 3 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\therefore \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

d) rank(A) = 3

nullity(A) = 2

5. (4 marks) Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ satisfying

$$T(\mathbf{i}) = \begin{bmatrix} -5 \\ 2 \end{bmatrix} \quad \text{and} \quad T(\mathbf{j}) = \begin{bmatrix} -7 \\ 3 \end{bmatrix}.$$

(a) Evaluate $T(2\mathbf{i} - 3\mathbf{j})$.

(b) Find the standard matrix for T^{-1}.

(c) Evaluate $T^{-1}(\mathbf{i})$ and $T^{-1}(\mathbf{j})$.

a) $T(2\hat{\imath} - 3\hat{\jmath}) = 2T(\hat{\imath}) - 3T(\hat{\jmath}) = 2\begin{bmatrix} -5 \\ 2 \end{bmatrix} - 3\begin{bmatrix} -7 \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \end{bmatrix}$

b) matrix for T is $\begin{bmatrix} -5 & -7 \\ 2 & 3 \end{bmatrix}$

$\therefore$ matrix for T^{-1} is $\begin{bmatrix} -5 & -7 \\ 2 & 3 \end{bmatrix}^{-1} = \frac{1}{-1}\begin{bmatrix} 3 & 7 \\ -2 & -5 \end{bmatrix} = \begin{bmatrix} -3 & -7 \\ 2 & 5 \end{bmatrix}$

c) $T^{-1}(\hat{\imath}) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and $T^{-1}(\hat{\jmath}) = \begin{bmatrix} -7 \\ 5 \end{bmatrix}$