

MATH 251 (Winter, 2019)
Term Test 1

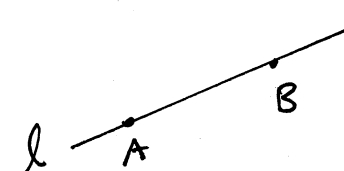
by George Ballinger

 Answer the questions in the space provided.
 This test has 5 questions for a total of 25 marks.

1. (6 marks) Consider the three points

$$A = (1, 2, 2), \quad B = (1, 0, 3), \quad C = (0, 2, 1).$$

- (a) Find parametric equations for the line passing through points A and B .
 (b) At what point does the line from part (a) intersect the xy -plane?
 (c) Find an equation in the general form $ax + by + cz = d$ for the plane containing the three points A , B and C .

a)  $\vec{d} = \vec{AB} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ $\vec{p} = \vec{A} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$$\vec{x} = \vec{p} + t\vec{d} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + t \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

$$\therefore \begin{cases} x = 1 \\ y = 2 - 2t \\ z = 2 + t \end{cases}$$

b) Intersection of xy -plane is where $z = 0$.
 $\therefore 2 + t = 0 \Rightarrow t = -2 \Rightarrow (x, y, z) = (1, 6, 0)$

c) $\vec{AC} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$ $\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -2 & 1 \\ -1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} \\ 0 & -2 \\ -1 & 0 \end{vmatrix}$

$$= (2\hat{i} - \hat{j}) - (2\hat{k}) = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

$$\therefore \vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$$

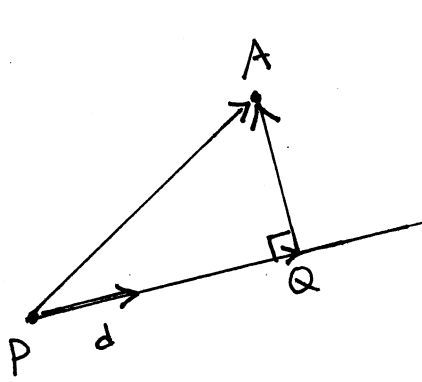
$$\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$2x - y - 2z = -4$$

2. (5 marks) Consider the line through the point $P = (3, -1, 2)$ with direction vector

$$\mathbf{d} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

Find the distance from the point $A = (-5, 4, 3)$ to the line. Give an exact answer.



$$\begin{aligned} \vec{PA} &= \begin{bmatrix} -8 \\ 5 \\ 1 \end{bmatrix} \\ \vec{PQ} &= \text{proj}_{\vec{d}}(\vec{PA}) \\ &= \frac{\vec{PA} \cdot \vec{d}}{\vec{d} \cdot \vec{d}} \vec{d} \\ &= \frac{10}{5} \vec{d} = 2\vec{d} = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \vec{QA} &= \vec{PA} - \vec{PQ} \\ &= \begin{bmatrix} -8 \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -6 \\ 5 \\ -3 \end{bmatrix} \end{aligned}$$

$$\therefore \text{distance} = \|\vec{QA}\| = \sqrt{36 + 25 + 9} = \sqrt{70}$$

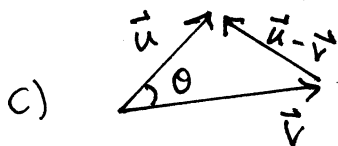
3. (5 marks) Consider the vectors

$$\mathbf{u} = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

- (a) Determine whether or not \mathbf{u} or \mathbf{v} are unit vectors.
 (b) Find the angle $0^\circ \leq \theta \leq 180^\circ$ between \mathbf{u} and \mathbf{v} . Round your answer to the nearest degree.
 (c) Find the area of the triangle formed by \mathbf{u} , \mathbf{v} and $\mathbf{u} - \mathbf{v}$. Round your answer to two decimal places.

$$\left. \begin{aligned} \text{a) } \|\vec{u}\| &= \sqrt{4+25+9} = \sqrt{38} \neq 1 \\ \|\vec{v}\| &= \sqrt{1+1+1} = \sqrt{3} \neq 1 \end{aligned} \right\} \therefore \text{Not unit vectors}$$

$$\text{b) } \cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \frac{-4}{\sqrt{38} \sqrt{3}} \Rightarrow \theta = \cos^{-1}\left(\frac{-4}{\sqrt{38} \sqrt{3}}\right) \approx 112^\circ$$



$$\text{Area} = \frac{1}{2} \|\vec{u} \times \vec{v}\|$$

$$\begin{aligned} \text{Area} &= \frac{1}{2} \|\vec{u}\| \|\vec{v}\| \sin \theta \\ &= \frac{1}{2} \sqrt{38} \sqrt{3} \sin 112^\circ \\ &\approx 4.95 \end{aligned}$$

(For exact value, use identity $\sin \theta = \sqrt{1 - \cos^2 \theta}$)

$$\begin{aligned} &= \sqrt{1 - \left(\frac{-4}{\sqrt{38} \sqrt{3}}\right)^2} = \sqrt{1 - \frac{16}{38 \cdot 3}} \\ &= \sqrt{\frac{98}{38 \cdot 3}}, \text{ so that} \\ \text{Area} &= \frac{1}{2} \sqrt{38} \sqrt{3} \cdot \sqrt{\frac{98}{38 \cdot 3}} = \frac{\sqrt{98}}{2} \end{aligned}$$

OR

$$\begin{aligned} \vec{u} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 5 & 3 \\ 1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} \\ -2 & 5 \\ 1 & -1 \end{vmatrix} \\ &= (5\hat{i} + 3\hat{j} + 2\hat{k}) - (5\hat{k} - 3\hat{i} - 2\hat{j}) \\ &= 8\hat{i} + 5\hat{j} - 3\hat{k} \\ \therefore \text{Area} &= \frac{1}{2} \sqrt{64 + 25 + 9} \\ &= \frac{\sqrt{98}}{2} \approx 4.95 \end{aligned}$$

4. (5 marks) Use the Gauss-Jordan Elimination method to find all the solutions of the system of linear equations. Write your answer in column vector form.

$$\begin{cases} x_1 - x_2 - x_3 + 2x_4 = 1 \\ 2x_1 - 2x_2 - x_3 + 3x_4 = 3 \\ -x_1 + x_2 - x_3 = -3 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{array} \right] \xrightarrow{\substack{R_2 - 2R_1 \\ R_3 + R_1}} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 2 & -2 \end{array} \right]$$

$$\xrightarrow{\substack{R_1 + R_2 \\ R_3 + 2R_2}} \left[\begin{array}{cccc|c} \textcircled{1} & -1 & 0 & 1 & 2 \\ 0 & 0 & \textcircled{1} & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} x_1, x_3 \text{ leading} \\ x_2, x_4 \text{ free} \end{array}$$

Let $x_2 = s$, $x_4 = t$.

$$\text{Then } x_1 - x_2 + x_4 = 2 \implies x_1 = 2 + x_2 - x_4 = 2 + s - t$$

$$x_3 - x_4 = 1 \implies x_3 = 1 + x_4 = 1 + t$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

5. (4 marks) Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ k \end{bmatrix}, \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}.$$

For what value(s) of k (if any) is \mathbf{u} in $\text{span}(\mathbf{v}_1, \mathbf{v}_2)$?

\vec{u} is in $\text{span}(\vec{v}_1, \vec{v}_2)$ iff $C_1 \vec{v}_1 + C_2 \vec{v}_2 = \vec{u}$

for constants C_1, C_2

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 2 & k & 5 \end{array} \right] \xrightarrow{\substack{R_2 - 2R_1 \\ R_3 - 2R_1}} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & k & 1 \end{array} \right] \xrightarrow{R_3 - kR_2} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1+k \end{array} \right]$$

Need $1+k=0$, i.e. $k=-1$
for there to be a
solution ($C_1=2, C_2=-1$)

$\therefore k$ must be -1 (in which case $\vec{u} = 2\vec{v}_1 - \vec{v}_2$)