CAMOSUN

Name: SOLUTIONS

MATH 251 (Winter, 2019)
Term Test 1
by George Ballinger

Answer the questions in the space provided. This test has 5 questions for a total of 25 marks.

1. (6 marks) Consider the three points

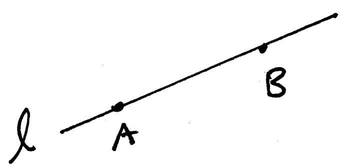
$$A = (1, 2, 2), \quad B = (1, 0, 3), \quad C = (0, 2, 1).$$

(a) Find parametric equations for the line passing through points A and B.

(b) At what point does the line from part (a) intersect the xy-plane?

(c) Find an equation in the general form $ax + by + cz = d$ for the plane containing the three points A, B and C.

a)



$$\vec{d} = \overrightarrow{AB} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \qquad \vec{p} = \vec{A} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\vec{x} = \vec{p} + t\vec{d} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + t \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

$$\therefore \begin{cases} x = 1 \\ y = 2 - 2t \\ z = 2 + t \end{cases}$$

b) Intersection of xy-plane is where $z = 0$.

$\therefore\ 2 + t = 0 \Rightarrow t = -2 \Rightarrow (x, y, z) = (1, 6, 0)$

c) $$\overrightarrow{AC} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \qquad \vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & -2 & 1 \\ -1 & 0 & -1 \end{vmatrix} \begin{matrix} \hat{\imath} & \hat{\jmath} \\ 0 & -2 \\ -1 & 0 \end{matrix}$$

$$= (2\hat{\imath} - \hat{\jmath}) - (2\hat{k}) = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

$$\therefore\ \vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$$

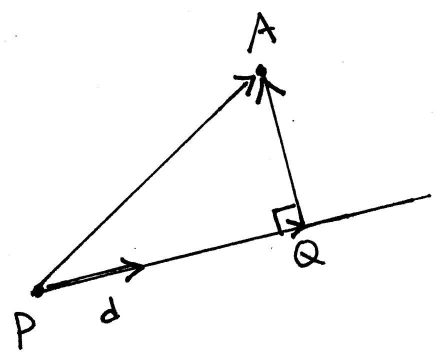
$$\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$2x - y - 2z = -4$$

2. (5 marks) Consider the line through the point $P = (3, -1, 2)$ with direction vector

$$\mathbf{d} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

Find the distance from the point $A = (-5, 4, 3)$ to the line. Give an exact answer.



$$\vec{PA} = \begin{bmatrix} -8 \\ 5 \\ 1 \end{bmatrix}$$

$$\vec{PQ} = \text{proj}_{\vec{d}}(\vec{PA}) = \frac{\vec{PA} \cdot \vec{d}}{\vec{d} \cdot \vec{d}} \vec{d} = \frac{10}{5} \vec{d} = 2\vec{d} = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$$

$$\vec{QA} = \vec{PA} - \vec{PQ} = \begin{bmatrix} -8 \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -6 \\ 5 \\ -3 \end{bmatrix}$$

$\therefore$ distance $= \|\vec{QA}\| = \sqrt{36 + 25 + 9} = \sqrt{70}$

3. (5 marks) Consider the vectors

$$\mathbf{u} = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

(a) Determine whether or not $\mathbf{u}$ or $\mathbf{v}$ are unit vectors.

(b) Find the angle $0° \leq \theta \leq 180°$ between $\mathbf{u}$ and $\mathbf{v}$. Round your answer to the nearest degree.

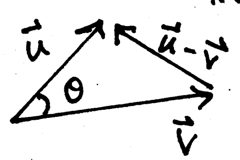
(c) Find the area of the triangle formed by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$. Round your answer to two decimal places.

a) $\|\vec{u}\| = \sqrt{4+25+9} = \sqrt{38} \neq 1$

$\|\vec{v}\| = \sqrt{1+1+1} = \sqrt{3} \neq 1$

$\therefore$ Not unit vectors

b) $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \dfrac{-4}{\sqrt{38}\sqrt{3}} \Rightarrow \theta = \cos^{-1}\left(\dfrac{-4}{\sqrt{38}\sqrt{3}}\right) \approx 112°$

c) 

$\text{Area} = \frac{1}{2}\|\vec{u}\times\vec{v}\|$

$\text{Area} = \frac{1}{2}\|\vec{u}\|\,\|\vec{v}\|\sin\theta$

$= \frac{1}{2}\sqrt{38}\sqrt{3}\ \sin 112°$

≈ 4.95

(For exact value, use identity $\sin\theta = \sqrt{1-\cos^2\theta}$

$= \sqrt{1-\left(\frac{-4}{\sqrt{38}\sqrt{3}}\right)^2} = \sqrt{1-\frac{16}{38\cdot 3}}$

$= \sqrt{\frac{98}{38\cdot 3}}$, so that

$\text{Area} = \frac{1}{2}\sqrt{38}\sqrt{3}\cdot\sqrt{\frac{98}{38\cdot 3}} = \frac{7\sqrt{2}}{2}$.)

OR

$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 5 & 3 \\ 1 & -1 & 1 \end{vmatrix} \begin{matrix} \hat{i} & \hat{j} \\ -2 & 5 \\ 1 & -1 \end{matrix}$

$= (5\hat{i} + 3\hat{j} + 2\hat{k}) - (5\hat{k} - 3\hat{i} - 2\hat{j})$

$= 8\hat{i} + 5\hat{j} - 3\hat{k}$

$\therefore \text{Area} = \frac{1}{2}\sqrt{64+25+9} = \frac{\sqrt{98}}{2}$

$= \frac{7\sqrt{2}}{2} \approx 4.95$

4. (5 marks) Use the Gauss-Jordan Elimination method to find all the solutions of the system of linear equations. Write your answer in column vector form.

$$\begin{cases} x_1 - x_2 - x_3 + 2x_4 = 1 \\ 2x_1 - 2x_2 - x_3 + 3x_4 = 3 \\ -x_1 + x_2 - x_3 = -3 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{array}\right] \xrightarrow{\substack{R_2-2R_1 \\ R_3+R_1}} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 2 & -2 \end{array}\right]$$

$$\xrightarrow{\substack{R_1+R_2 \\ R_3+2R_2}} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_1, x_3 leading
x_2, x_4 free

Let $x_2 = s$, $x_4 = t$.

Then $x_1 - x_2 + x_4 = 2 \Longrightarrow x_1 = 2 + x_2 - x_4 = 2 + s - t$

$x_3 - x_4 = 1 \Longrightarrow x_3 = 1 + x_4 = 1 + t$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

5. (4 marks) Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix}1\\2\\2\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}0\\1\\k\end{bmatrix}, \quad \text{and} \quad \mathbf{u} = \begin{bmatrix}2\\3\\5\end{bmatrix}.$$

For what value(s) of k (if any) is $\mathbf{u}$ in $\text{span}(\mathbf{v}_1, \mathbf{v}_2)$?

$\vec{u}$ is in $\text{span}(\vec{v}_1, \vec{v}_2)$ iff $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{u}$ for constants c_1, c_2

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 2 & K & 5 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2-2R_1} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & K & 1 \end{array}\right] \xrightarrow{R_3-KR_2} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1+K \end{array}\right]$$

Need $1+K=0$, i.e. $K=-1$ for there to be a solution ($c_1=2$, $c_2=-1$)

$\therefore$ K must be -1 (in which case $\vec{u} = 2\vec{v}_1 - \vec{v}_2$)