CAMOSUN

Name: SOLUTIONS

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MATH 251 (Winter, 2022)

Test 1

1. (6 marks) Consider the parallelogram formed by vectors $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -5 \\ 0 \end{bmatrix}$.

   (a) Find the area of the parallelogram. Give an exact, simplified answer.

   (b) Find the angle $0° \leq \theta \leq 180°$ between $\mathbf{u}$ and $\mathbf{v}$. Round your answer to two decimal places.

   (c) Find vectors representing each of the diagonals of the parallelogram.

a) $\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -2 & -1 \\ 2 & -5 & 0 \end{vmatrix} \begin{matrix} \hat{\imath} & \hat{\jmath} \\ 3 & -2 \\ 2 & -5 \end{matrix} = (-2\hat{\jmath} - 15\hat{k}) - (-4\hat{k} + 5\hat{\imath}) = -5\hat{\imath} - 2\hat{\jmath} - 11\hat{k}$

Area $= \|\vec{u} \times \vec{v}\| = \sqrt{(-5)^2 + (-2)^2 + (-11)^2} = \sqrt{150} = 5\sqrt{6}$

b) $\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \dfrac{16}{\sqrt{14}\sqrt{29}} \Rightarrow \theta = \cos^{-1}\left(\dfrac{16}{\sqrt{14}\sqrt{29}}\right) \approx 37.43°$

c) diagonals are $\vec{u} + \vec{v} = \begin{bmatrix} 5 \\ -7 \\ -1 \end{bmatrix}$ and $\underbrace{\vec{u} - \vec{v} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}}$

or $\vec{v} - \vec{u} = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$

2. (6 marks) Consider the points $A = (1, 2, 3)$, $B = (3, 5, -3)$ and $C = (11, 10, -6)$.

   (a) Find a unit vector that points in the direction of $\overrightarrow{AB}$.

   (b) Find an equation, in vector form, of the line passing through points A and B.

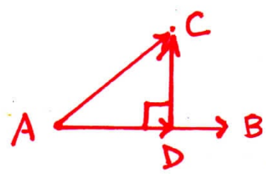
   (c) Find the distance between point C and the line in part (b).

a) $\overrightarrow{AB} = \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix}$ $\quad \|\overrightarrow{AB}\| = \sqrt{49} = 7$ $\quad \therefore$ unit vector is $\frac{1}{7}\begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix} = \begin{bmatrix} 2/7 \\ 3/7 \\ -6/7 \end{bmatrix}$

b) $\vec{x} = \vec{A} + t\,\overrightarrow{AB} \quad \Rightarrow \quad \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t\begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix}$

or $\vec{B}$

c)



$\overrightarrow{AC} = \begin{bmatrix} 10 \\ 8 \\ -9 \end{bmatrix}$

$\overrightarrow{AD} = \mathrm{proj}_{\overrightarrow{AB}}(\overrightarrow{AC}) = \left(\dfrac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\overrightarrow{AB}\cdot\overrightarrow{AB}}\right)\overrightarrow{AB} = \dfrac{98}{49}\overrightarrow{AB} = 2\overrightarrow{AB} = \begin{bmatrix} 4 \\ 6 \\ -12 \end{bmatrix}$

$\overrightarrow{DC} = \overrightarrow{AC} - \overrightarrow{AD} = \begin{bmatrix} 10 \\ 8 \\ -9 \end{bmatrix} - \begin{bmatrix} 4 \\ 6 \\ -12 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}$

$\therefore$ distance $= \|\overrightarrow{DC}\| = \sqrt{49} = 7$

3. (4 marks) Consider the line defined parametrically by

$$\begin{cases} x = & 7 - 2t \\ y = & 1 + 5t \\ z = & -6 + 3t. \end{cases}$$

(a) Find the coordinates of the point where the line intersects the xy-plane.

(b) Find an equation, in general form, of the plane passing through the point $P = (1, 2, -1)$ that is perpendicular to the given line.

a) $z = 0 \Rightarrow -6 + 3t = 0 \Rightarrow t = 2$

$\therefore x = 7 - 2(2) = 3$ and $y = 1 + 5(2) = 11$

So, point is $(3, 11, 0)$

b) direction vector of line, $\vec{d} = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$, is $\perp$ to the plane and is a normal vector to the plane, i.e. $\vec{n} = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$. Equation of plane is $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$ where $\vec{p} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$. Expanding gives $-2x + 5y + 3z = 5$

4. (5 marks) Use the Gauss-Jordan Elimination method to find all the solutions of the system of linear equations. Write your answer in column vector form. Clearly show your steps, including your row operations.

$$\begin{cases} \quad\;\; 2x_2 - 4x_3 - 12x_4 = 14 \\ x_1 \quad\;\; + 3x_3 + \;6x_4 = -9 \\ \quad\;\; 2x_2 - 3x_3 - 11x_4 = 9 \end{cases}$$

$$\left[\begin{array}{cccc|c} 0 & 2 & -4 & -12 & 14 \\ 1 & 0 & 3 & 6 & -9 \\ 0 & 2 & -3 & -11 & 9 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 1 & 0 & 3 & 6 & -9 \\ 0 & 2 & -4 & -12 & 14 \\ 0 & 2 & -3 & -11 & 9 \end{array}\right]$$

$$\xrightarrow[R_3 - R_2]{\frac{1}{2}R_2} \left[\begin{array}{cccc|c} 1 & 0 & 3 & 6 & -9 \\ 0 & 1 & -2 & -6 & 7 \\ 0 & 0 & 1 & 1 & -5 \end{array}\right] \xrightarrow[R_2 + 2R_3]{R_1 - 3R_3} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 3 & 6 \\ 0 & \boxed{1} & 0 & -4 & -3 \\ 0 & 0 & \boxed{1} & 1 & -5 \end{array}\right]$$

x_1, x_2, x_3 leading

x_4 free $\Rightarrow$ Let $x_4 = t$

$x_1 + 3x_4 = 6 \Rightarrow x_1 = -3t + 6$

$x_2 - 4x_4 = -3 \Rightarrow x_2 = 4t - 3$

$x_3 + x_4 = -5 \Rightarrow x_3 = -t - 5$

$$\therefore \vec{x} = \begin{bmatrix} 6 \\ -3 \\ -5 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 4 \\ -1 \\ 1 \end{bmatrix}$$

5. (2 marks) Set up, **but do not solve**, a homogenous system of linear equations that could be used to balance the following unbalanced chemical equation involving the elements Mn, O, H and Cl.

$$MnO_2 + HCl \to MnCl_2 + H_2O + Cl_2$$

$$x_1 MnO_2 + x_2 HCl \to x_3 MnCl_2 + x_4 H_2O + x_5 Cl_2$$

Mn $\quad x_1 = x_3$

O $\quad 2x_1 = x_4$

H $\quad x_2 = 2x_4$

Cl $\quad x_2 = 2x_3 + 2x_5$

$$\Rightarrow \left\{ \begin{array}{l} x_1 - x_3 = 0 \\ 2x_1 - x_4 = 0 \\ x_2 - 2x_4 = 0 \\ x_2 - 2x_3 - 2x_5 = 0 \end{array} \right.$$

6. (2 marks) Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 4 \\ -2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 5 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 4 \\ 5 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 7 \\ -9 \\ -2 \\ -4 \\ 9 \end{bmatrix}$$

and suppose

$$\left[\begin{array}{rrr|r} 1 & 2 & 0 & 7 \\ 3 & -1 & 4 & -9 \\ 4 & 3 & 5 & -2 \\ -2 & 5 & 2 & -4 \\ 3 & 2 & 1 & 9 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

If possible, express $\mathbf{v}_4$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. If not possible, explain why.

Need to find x_1, x_2, x_3 so that $x_1\vec{V_1} + x_2\vec{V_2} + x_3\vec{V_3} = \vec{V_4}$.

From RREF we see $x_1 = 3$, $x_2 = 2$, $x_3 = -4$.

$\therefore \vec{V_4} = 3\vec{V_1} + 2\vec{V_2} - 4\vec{V_3}$