Systems of Two Linear Equations in Two Variables

A system of two linear equations in two variables can have one of three possible sets of solutions. It can have a unique solution, no solution, or infinitely many solutions. The following examples illustrate these possibilities.

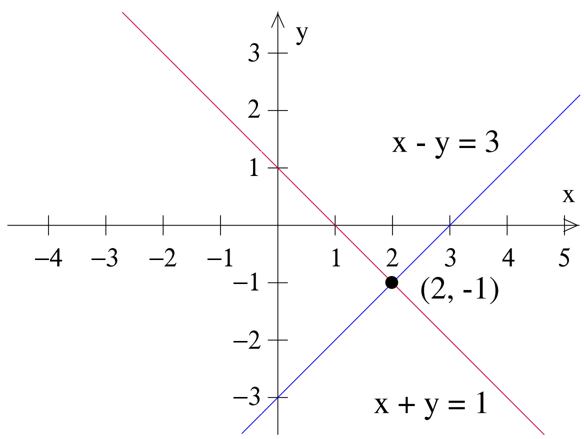
1. Unique Solution: Consider the system

$$\begin{cases} x + y = 1 \\ x - y = 3. \end{cases}$$

By adding these equations, we get $2x = 4$ and so $x = 2$. Substituting $x = 2$ into either of the equations yields $y = -1$. So, this system has a unique solution $(x, y) = (2, -1)$. Writing the solution in vector form gives

$$\mathbf{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Graphically, these equations represent intersecting lines in the plane and the point of intersection represents the unique solution.



2. No Solution: Consider the system

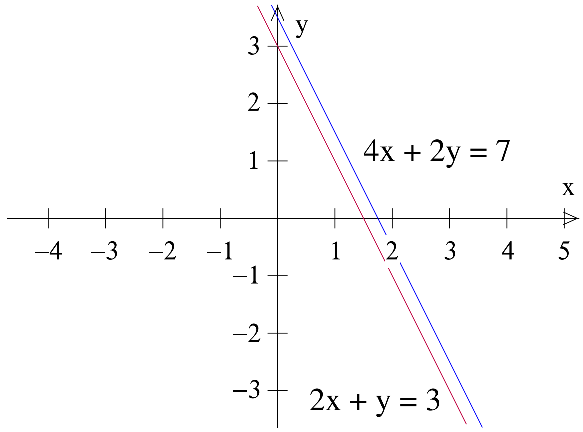
$$\begin{cases} 2x + y = 3 \\ 4x + 2y = 7. \end{cases}$$

Dividing the second equation by 2 gives $2x + y = 7/2$. Since $2x + y$ cannot equal both 3 from the first equation and $7/2$ from the second equation, then this sytem of equations has no solution. Its solution set is the empty set, $\varnothing$.

Rewriting these equations in slope-intercept from gives

$$\begin{cases} y = -2x + 3 \\ y = -2x + 7/2. \end{cases}$$

These equations represent parallel lines having the same slope of -2 but different y-intercepts. Since there are no points of intersection, then there is no solution.



2. Infinitely Many Solutions: Consider the system

$$\begin{cases} 2x + y = 3 \\ 4x + 2y = 6. \end{cases}$$

Dividing the second equation by 2 gives $2x + y = 3$, which is the same as the first equation. These two equations are therefore equivalent and represent the same line, which in slope-intercept form can be written $y = -2x + 3$. Therefore, there are infinitely many solutions consisting of all points (x, y) lying on this common, overlapping line.

The solutions can be written in vector form by introducing a parameter t and letting $x = t$ and $y = -2t + 3$. Solution vectors are therefore

$$\mathbf{x} = \begin{bmatrix} t \\ -2t + 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \text{ for any } t \in \mathbb{R}.$$