Name: SOLUTIONS

Mark: $\overline{}$ **25**

MATH 251 (Winter, 2024)

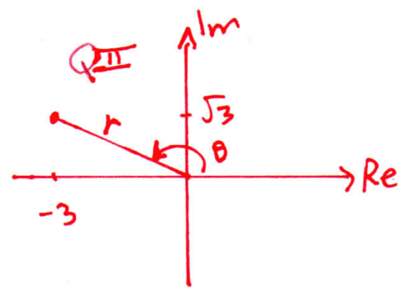
Test 3

1. (4 marks) Evaluate $4\angle 120^\circ - 2\angle 60^\circ$. Express your answer in phasor form, $r\angle\theta$, using exact values for both r and θ (i.e. no decimals).

$$4\angle 120^\circ - 2\angle 60^\circ = (4\cos 120^\circ + 4i\sin 120^\circ) - (2\cos 60^\circ + 2i\sin 60^\circ)$$

$$= \left(4\left(-\tfrac{1}{2}\right) + 4i\left(\tfrac{\sqrt{3}}{2}\right)\right) - \left(2\left(\tfrac{1}{2}\right) + 2i\left(\tfrac{\sqrt{3}}{2}\right)\right)$$

$$= (-2 + 2i\sqrt{3}) - (1 + i\sqrt{3}) = -3 + i\sqrt{3}$$



$$r = \sqrt{(-3)^2 + (\sqrt{3})^2} = \sqrt{9+3} = \sqrt{12} = 2\sqrt{3}$$

$$\tan\theta = \frac{\sqrt{3}}{-3} \implies \theta = 150^\circ \text{ in QII}$$

$$\therefore \quad 2\sqrt{3}\angle 150^\circ$$

2. (6 marks) Let $A = \begin{bmatrix} 9 & -4 \\ 5 & 1 \end{bmatrix}$.

   (a) Find the area of the parallelogram formed by the column vectors of A.

   (b) Is $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ an eigenvector of A? Briefly explain.

   (c) Find the eigenvalues of A.

a) Area $= |\det(A)| = |29| = 29$

b) $A\vec{x} = \begin{bmatrix} 9 & -4 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix} \leftarrow$ not a scalar multiple of $\vec{x}$

$\therefore$ $\vec{x}$ is <u>NOT</u> an eigenvector of A

c) $\det(A - \lambda I) = \begin{vmatrix} 9-\lambda & -4 \\ 5 & 1-\lambda \end{vmatrix} = (9-\lambda)(1-\lambda) + 20$

$= \lambda^2 - 10\lambda + 29 = (\lambda-5)^2 + 4 = 0 \Longrightarrow (\lambda - 5)^2 = -4$

$\therefore \lambda = 5 \pm 2i$

3. (5 marks) Let $A = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) Find the eigenvalues and corresponding eigenspaces of A.

(b) If possible, find a diagonal matrix D and an invertible matrix P so that $P^{-1}AP = D$. If it's not possible, then explain why.

a) $\lambda = 1$ (alg. mult. 3) is the only eigenvalue (diagonal entries of upper triangular matrix)

$A - \lambda I = \begin{bmatrix} 0 & 2 & 6 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ Solve $(A - \lambda I)\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccc|c} 0 & 2 & 6 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\substack{\frac{1}{2}R_1 \\ \frac{1}{2}R_2}} \left[\begin{array}{ccc|c} 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 - 3R_2} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = t$ (free), $x_2 = 0$, $x_3 = 0$

$\therefore \vec{x} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\therefore E_\lambda = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$

b) Not possible since there is only 1 (not 3) L.I. eigenvectors. (A is defective; geo. mult. of $\lambda = 1$ is less than alg. mult.)

4. (4 marks) Use Cramer's Rule to solve for x_2 in the system of linear equations $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 0 & 1 & 5 & 0 \\ 3 & 1 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 8 \\ 9 \\ 2 \\ 7 \end{bmatrix}.$$

$$\det(A) = -3\begin{vmatrix} 1 & 5 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{vmatrix} = (-3)(1)\begin{vmatrix} 1 & 5 \\ 0 & 2 \end{vmatrix} = (-3)(1)(2) = -6$$

$$\det(A_2(\vec{b})) = \begin{vmatrix} 0 & 8 & 5 & 0 \\ 3 & 9 & 0 & 3 \\ 0 & 2 & 2 & 0 \\ 0 & 7 & 0 & 1 \end{vmatrix} = (-3)\begin{vmatrix} 8 & 5 & 0 \\ 2 & 2 & 0 \\ 7 & 0 & 1 \end{vmatrix} = (-3)(1)\begin{vmatrix} 8 & 5 \\ 2 & 2 \end{vmatrix} = (-3)(1)(6) = -18$$

$$\therefore x_2 = \frac{\det(A_2(\vec{b}))}{\det(A)} = \frac{-18}{-6} = 3$$

5. (6 marks) Let

$$\mathbf{v}_1 = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix},$$

and note that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis for the subspace $W = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$ of $\mathbb{R}^4$.

(a) Is $\mathcal{B}$ an orthonormal set? Briefly explain.

(b) What vectors, if any, belong to both W and $W^{\perp}$?

(c) Find $\text{proj}_W(\mathbf{v})$ and $\text{perp}_W(\mathbf{v})$.

a) Yes since $\vec{v}_1 \cdot \vec{v}_2 = 0$, $\|\vec{v}_1\| = 1$ and $\|\vec{v}_2\| = 1$.

b) Only $\vec{0}$.

c) $\text{proj}_W(\vec{v}) = (\vec{v}\cdot\vec{v}_1)\vec{v}_1 + (\vec{v}\cdot\vec{v}_2)\vec{v}_2 = 5\vec{v}_1 + 3\vec{v}_2$

$$= \begin{bmatrix} -5/2 \\ 5/2 \\ 5/2 \\ 5/2 \end{bmatrix} + \begin{bmatrix} 3/2 \\ 3/2 \\ -3/2 \\ 3/2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 1 \\ 4 \end{bmatrix}$$

and $\text{perp}_W(\vec{v}) = \vec{v} - \text{proj}_W(\vec{v}) = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 3 \\ 1 \end{bmatrix}$.