CAMOSUN

Name: SOLUTIONS

Mark: $\overline{}$ **25**

MATH 251 (Winter, 2024)

Test 2

1. (3 marks) Evaluate $A^{-1} + 2A^T - 4I$, where $A = \begin{bmatrix} -8 & 5 \\ -3 & 2 \end{bmatrix}$.

$$A^{-1} + 2A^T - 4I = \frac{1}{-1}\begin{bmatrix} 2 & -5 \\ 3 & -8 \end{bmatrix} + 2\begin{bmatrix} -8 & -3 \\ 5 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 5 \\ -3 & 8 \end{bmatrix} + \begin{bmatrix} -16 & -6 \\ 10 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -22 & -1 \\ 7 & 8 \end{bmatrix}$$

2. (3 marks) Let S be the set of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ such that $|x| = |y| = |z|$.

   (a) Give examples of two linearly independent vectors, $\mathbf{u}$ and $\mathbf{v}$, belonging to S.

   (b) Use your vectors from part (a) to prove that S is not a subspace of $\mathbb{R}^3$.

a) $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

b) $\vec{u} + \vec{v} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \notin S$ since $|2| \neq |0|$

S is not closed under vector addition.

$\therefore$ S is not a subspace of $\mathbb{R}^3$.

3. (6 marks) Let $A = \begin{bmatrix} 2 & -3 \\ -6 & 5 \end{bmatrix}$.

   (a) Find an LU factorization of A.

   (b) Use the LU method to solve the system

$$\begin{cases} 2x_1 - 3x_2 = -7 \\ -6x_1 + 5x_2 = 1. \end{cases}$$

a) $A = \begin{bmatrix} 2 & -3 \\ -6 & 5 \end{bmatrix} \xrightarrow{R_2+3R_1} \begin{bmatrix} 2 & -3 \\ 0 & -4 \end{bmatrix} = U$ where $L = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

$\therefore A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 0 & -4 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -7 \\ 1 \end{bmatrix}$ $\quad y_1 = -7$

$-3y_1 + y_2 = 1 \Rightarrow 21 + y_2 = 1 \Rightarrow y_2 = -20$

$\begin{bmatrix} 2 & -3 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -7 \\ -20 \end{bmatrix}$ $\quad -4x_2 = -20 \Rightarrow x_2 = 5$

$2x_1 - 3x_2 = -7 \Rightarrow 2x_1 - 15 = -7$

$\Rightarrow 2x_1 = 8 \Rightarrow x_1 = 4$

$\therefore \vec{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

4. (4 marks) Let $A = \begin{bmatrix} 0 & a & 0 \\ 1 & 0 & 0 \\ b & 0 & 1 \end{bmatrix}$, where a and b are scalars and $a \neq 0$.

   (a) Use the Gauss-Jordan method to find A^{-1}.

   (b) Find a and b if A is symmetric.

a) $\left[\begin{array}{ccc|ccc} 0 & a & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ b & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & a & 0 & 1 & 0 & 0 \\ b & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

$\xrightarrow{\substack{\frac{1}{a}R_2 \\ R_3 - bR_1}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1/a & 0 & 0 \\ 0 & 0 & 1 & 0 & -b & 1 \end{array}\right] \quad \therefore A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1/a & 0 & 0 \\ 0 & -b & 1 \end{bmatrix}$

b) $A = A^T \Rightarrow a = 1$ and $b = 0$.

5. (5 marks) Suppose

$$A = \begin{bmatrix} 4 & 8 & 1 & 9 & 5 \\ 3 & 6 & 2 & 8 & 4 \\ 2 & 4 & 3 & 7 & 3 \\ 1 & 2 & 4 & 6 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find a basis $\mathcal{B}$ for the column space of A consisting of columns of A.

(b) For each column vector $\mathbf{v}$ of A that is *not* in your basis $\mathcal{B}$ from part (a), find its coordinate vector with respect to the basis $\mathcal{B}$, i.e. find $[\mathbf{v}]_{\mathcal{B}}$.

(c) Find rank(A), rank(A^T), nullity(A) and nullity(A^T).

a) $B = \left\{ \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 3 \\ 3 \end{bmatrix} \right\}$

b) $\vec{v}_2 = \begin{bmatrix} 8 \\ 6 \\ 4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} = 2\vec{v}_1 \quad \therefore \ [\vec{v}_2]_B = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

$\vec{v}_4 = \begin{bmatrix} 9 \\ 8 \\ 7 \\ 6 \end{bmatrix} = 2 \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = 2\vec{v}_1 + \vec{v}_3 \quad \therefore \ [\vec{v}_4]_B = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

c) rank $(A) = 3$ (dim B)

rank $(A^T) =$ rank $(A) = 3$

nullity $(A) =$ # columns of A − rank $(A) = 5 - 3 = 2$

nullity $(A^T) =$ # columns of A^T − rank $(A^T) = 4 - 3 = 1$

6. (4 marks) Suppose $S : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation defined by

$$S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 4y \\ 2x \end{bmatrix}.$$

(a) What is the standard matrix for S?

(b) Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates a vector $60°$ counter-clockwise and then performs the transformation S. Find the standard matrix for T. Give an exact, simplified answer (no decimals).

a) $[S] = \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}$

b) $[T] = [S][R_{60°}] = \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix}$

$= \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 2\sqrt{3} & 2 \\ 1 & -\sqrt{3} \end{bmatrix}$