CAMOSUN

Name: SOLUTIONS

Mark: $\overline{\quad}$ **25**

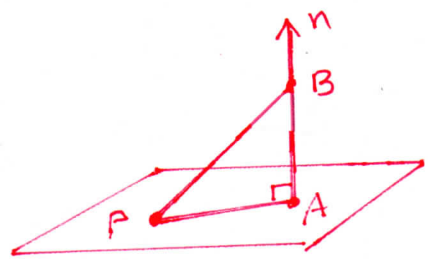
MATH 251 (Winter, 2024)

Test 1

1. (6 marks) Using projections, find the distance between point $B = (-3, -4, 7)$ and the plane $2x + y - 2z = 3$ and find the coordinates of the point A in the plane that is closest to point B.

$P = (0,3,0)$ is a point on the plane

$\vec{n} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ is a normal vector $\perp$ to the plane



$\vec{PB} = \begin{bmatrix} -3 \\ -7 \\ 7 \end{bmatrix}$

$$\vec{AB} = \text{proj}_{\vec{n}}(\vec{PB}) = \left(\frac{\vec{n} \cdot \vec{PB}}{\vec{n} \cdot \vec{n}}\right)\vec{n} = \frac{-27}{9}\vec{n} = -3\vec{n} = \begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix}$$

$$\text{distance} = \|\vec{AB}\| = \sqrt{(-6)^2 + (-3)^2 + 6^2} = \sqrt{81} = 9$$

$$\vec{AB} = \vec{B} - \vec{A} \Rightarrow \vec{A} = \vec{B} - \vec{AB} = \begin{bmatrix} -3 \\ -4 \\ 7 \end{bmatrix} - \begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \quad \therefore A = (3, -1, 1)$$

2. Find the equation of the plane containing the points $A = (1, 2, 3)$, $B = (2, 0, 3)$ and $C = (4, 1, 2)$, and write your answer in

(a) (3 marks) vector form.

$\vec{AB} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ and $\vec{AC} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$ are direction vectors

$\therefore \ \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$

using point A

(b) (3 marks) general form.

$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 0 \\ 3 & -1 & -1 \end{vmatrix} \begin{matrix} \hat{i} & \hat{j} \\ 1 & -2 \\ 3 & -1 \end{matrix} = (2\hat{i} - \hat{k}) - (-6\hat{k} - \hat{j}) = 2\hat{i} + \hat{j} + 5\hat{k}$

$= \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$

$\therefore \ \vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{A}$, where $\vec{A} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$2x + y + 5z = 19$

3. (a) (4 marks) Solve the system of linear equations by using the **Gauss-Jordan Elimination method**. Write your answer in column vector form. Clearly show your steps, including your row operations.

$$\begin{cases} x + 3y - 3z = 0 \\ x \qquad\;\; + 3z = 0 \\ 2x - \;\; y + 8z = 0 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -3 & 0 \\ 1 & 0 & 3 & 0 \\ 2 & -1 & 8 & 0 \end{array}\right] \rightarrow \begin{array}{l} \\ R_2-R_1 \\ R_3-2R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & -3 & 0 \\ 0 & -3 & 6 & 0 \\ 0 & -7 & 14 & 0 \end{array}\right] \rightarrow \begin{array}{l} \\ -\frac{1}{3}R_2 \\ -\frac{1}{7}R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

$$\rightarrow \begin{array}{l} R_1-3R_2 \\ \\ R_3-R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} x=-3t \\ y=2t \\ z=t \end{array} \qquad \therefore \vec{x} = t\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}, \; t \in \mathbb{R}$$

(b) (2 marks) Using your answer from part (a) and by referring to the definition of linear dependence, briefly explain why, or why not, the following vectors are linearly dependent.

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \; \mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \; \mathbf{w} = \begin{bmatrix} -3 \\ 3 \\ 8 \end{bmatrix}$$

$\vec{u}, \vec{v}$ and $\vec{w}$ are L.D. since $-3\vec{u} + 2\vec{v} + \vec{w} = \vec{0}$,

i.e. there are scalars $c_1 = -3$, $c_2 = 2$ and $c_3 = 1$,

not all 0, such that $c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = \vec{0}$.

4. Find the value(s) of k (if any) for which $\begin{bmatrix} k \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ k+4 \end{bmatrix}$ are

   (a) (2 marks) orthogonal vectors.

$$\begin{bmatrix} k \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ k+4 \end{bmatrix} = 0$$

$$-3k + k + 4 = 0$$

$$-2k = -4$$

$$k = 2$$

   (b) (3 marks) distinct parallel vectors.

$$\begin{bmatrix} -3 \\ k+4 \end{bmatrix} = c \begin{bmatrix} k \\ 1 \end{bmatrix}$$

$-3 = ck$ and $k+4 = c$

$$\therefore \; -3 = (k+4)k$$

$$-3 = k^2 + 4k$$

$$k^2 + 4k + 3 = 0$$

$$(k+3)(k+1) = 0$$

$k = -3$ or $k = -1$

↑

discard since vectors would be the same $\qquad \therefore k = -1$

5. (2 marks) Set up, **but do not solve**, a *homogeneous* system of linear equations that could be used to balance the following unbalanced chemical equation involving the elements C, O and H.

$$CO_2 + H_2O \rightarrow C_6H_{12}O_6 + O_2$$

$$x_1 CO_2 + x_2 H_2O \rightarrow x_3 C_6H_{12}O_6 + x_4 O_2$$

C	$x_1 = 6x_3$		$x_1 - 6x_3 = 0$
O	$2x_1 + x_2 = 6x_3 + 2x_4$	$\Rightarrow$	$2x_1 + x_2 - 6x_3 - 2x_4 = 0$
H	$2x_2 = 12x_3$		$2x_2 - 12x_3 = 0$