Subspaces and Bases

Definition: A **subspace** of $\mathbb{R}^n$ is any collection S of vectors in $\mathbb{R}^n$ such that

1. The zero vector $\mathbf{0}$ is in S.
2. If $\mathbf{u}$ and $\mathbf{v}$ are in S, then $\mathbf{u}+\mathbf{v}$ is in S. (S is **closed under addition**.)
3. If $\mathbf{u}$ is in S and c is a scalar, then $c\mathbf{u}$ is in S. (S is **closed under scalar multiplication**.)

Definition: Let A be an $m \times n$ matrix.

1. The **row space** of A is the subspace row(A) of $\mathbb{R}^n$ spanned by the rows of A.
2. The **column space** (also called **range space**) of A is the subspace col(A) of $\mathbb{R}^m$ spanned by the columns of A. Equivalently, col(A)=$\{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\}$.
3. The **null space** (also called **kernel**) of A is the subspace null(A) of $\mathbb{R}^n$ consisting of all solutions of the homogenous linear system $A\mathbf{x} = \mathbf{0}$; i.e. null(A)=$\{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$.

Definition: A **basis** for a subspace S of $\mathbb{R}^n$ is a set of vectors in S that

1. spans S and
2. is linearly independent.

The number of vectors in a basis for S is called the **dimension** of S, denoted dim S. Since $S = \{\mathbf{0}\}$ has no basis, we define $\dim\{\mathbf{0}\} = 0$.

Definition: The **rank** of a matrix A is the dimension of its row and column spaces, denoted rank(A).

Definition: The **nullity** of a matrix A is the dimension of its null space, denoted nullity(A).

Definition: Let S be a subspace of $\mathbb{R}^n$ and let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a basis for S. Let $\mathbf{v}$ be a vector in S, and write $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$. Then $c_1, c_2, \ldots, c_k$ are called the **coordinates of v with respect to $\mathcal{B}$**, and the column vector

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

is called the **coordinate vector of v with respect to $\mathcal{B}$**.