Properties of Vectors

Theorem 1.1: Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^n$ and let c and d be scalars. Then

(a) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(b) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

(c) $\mathbf{u}+\mathbf{0}=\mathbf{u}$

(d) $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$

(e) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

(f) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

(g) $c(d\mathbf{u})=(cd)\mathbf{u}$

(h) $1\mathbf{u}=\mathbf{u}$

Theorem 1.2: Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^n$ and let c be a scalar. Then

(a) $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$

(b) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$

(c) $(c\mathbf{u})\cdot\mathbf{v}=c(\mathbf{u}\cdot\mathbf{v})$

(d) $\mathbf{u}\cdot\mathbf{u}\geq 0$ and $\mathbf{u}\cdot\mathbf{u}=0$ if and only if $\mathbf{u}=\mathbf{0}$

Theorem 1.3: Let $\mathbf{v}$ be a vector in $\mathbb{R}^n$ and let c be a scalar. Then

(a) $\|\mathbf{v}\|=0$ if and only if $\mathbf{v}=\mathbf{0}$

(b) $\|c\mathbf{v}\|=|c|\|\mathbf{v}\|$

Theorem 1.4 (Cauchy-Schwarz Inequality): For all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$,

$$|\mathbf{u}\cdot\mathbf{v}|\leq\|\mathbf{u}\|\,\|\mathbf{v}\|$$

Theorem 1.5 (Triangle Inequality): For all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$,

$$\|\mathbf{u}+\mathbf{v}\|\leq\|\mathbf{u}\|+\|\mathbf{v}\|$$

Theorem 1.6 (Pythagoras' Theorem): For all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, $\|\mathbf{u}+\mathbf{v}\|^2=\|\mathbf{u}\|^2+\|\mathbf{v}\|^2$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.