Orthogonality

Definition: A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in $\mathbb{R}^n$ is called an **orthogonal set** if all pairs of distinct vectors in the set are orthogonal. In other words,

$$\mathbf{v}_i \cdot \mathbf{v}_j = 0 \quad \text{for all } i \neq j, \text{ where } i, j = 1, 2, \ldots, k$$

Definition: An **orthogonal basis** for a subspace W of $\mathbb{R}^n$ is a basis of W that is an orthogonal set.

Definition: A set of vectors in $\mathbb{R}^n$ is an **orthonormal set** if it is an orthogonal set of unit vectors. An **orthonormal basis** for a subspace W of $\mathbb{R}^n$ is a basis of W that is an orthonormal set.

Definition: An $n \times n$ matrix Q whose columns form an orthonormal set is called an **orthogonal matrix**.

Definition: Let W be a subspace of $\mathbb{R}^n$. A vector $\mathbf{v}$ in $\mathbb{R}^n$ is said to be **orthogonal to** W if $\mathbf{v}$ is orthogonal to every vector in W. The set of all vectors that are orthogonal to W is called the **orthogonal complement of** W, denoted $W^\perp$. In other words,

$$W^\perp = \{\mathbf{v} \text{ in } \mathbb{R}^n | \, \mathbf{v} \cdot \mathbf{w} = 0 \quad \text{for all } \mathbf{w} \text{ in } W\}$$

Definition: Let W be a subspace of $\mathbb{R}^n$ and let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be an orthogonal basis for W. For any vector $\mathbf{v}$ in $\mathbb{R}^n$, the **orthogonal projection of v onto** W is defined as

$$\begin{aligned}
\text{proj}_W(\mathbf{v}) &= \text{proj}_{\mathbf{u}_1}(\mathbf{v}) + \text{proj}_{\mathbf{u}_2}(\mathbf{v}) + \cdots + \text{proj}_{\mathbf{u}_k}(\mathbf{v}) \\
&= \left(\frac{\mathbf{u}_1 \cdot \mathbf{v}}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1 + \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}}{\mathbf{u}_2 \cdot \mathbf{u}_2}\right)\mathbf{u}_2 + \cdots + \left(\frac{\mathbf{u}_k \cdot \mathbf{v}}{\mathbf{u}_k \cdot \mathbf{u}_k}\right)\mathbf{u}_k
\end{aligned}$$

The **component of v orthogonal to** W is the vector

$$\text{perp}_W(\mathbf{v}) = \mathbf{v} - \text{proj}_W(\mathbf{v})$$