# MATH 251
# Practice Questions

1. Consider the following two vectors in $\mathbb{R}^2$.

$$\mathbf{u} = \begin{bmatrix} x \\ 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

Find the values of $x$ if

(a) $\mathbf{u}$ is parallel to $\mathbf{v}$, (b) $\mathbf{u}$ is perpendicular to $\mathbf{v}$, (c) the angle between $\mathbf{u}$ and $\mathbf{v}$ is $30°$.

*Answers:* (a) $x = 8/3$ (b) $x = -3/2$ (c) $x_1 \approx 16.6$ and $x_2 \approx 0.854$

2. (a) Find parametric equations for the line passing through the points

$$A = (2, 4, -1), \quad \text{and} \quad B = (5, 0, 7).$$

(b) At which point does the line intersect the $xy$-plane?

(c) Find the distance between the line in (a) and point $P = (1, 0, 1)$.

*Answers:* (a) $x = 2 + 3t,\ y = 4 - 4t,\ z = -1 + 8t$ (b) $(19/8, 7/2, 0)$ (c) $\approx 3.398$

3. Consider the four points

$$A = (0, 1, 0), \quad B = (3, 2, 0), \quad C = (0, 3, -3), \quad \text{and} \quad P = (1, 1, 2).$$

(a) Find an equation in general form for the plane passing through $A$, $B$, and $C$.

(b) Find the distance between the plane in (a) and point $P$.

(c) Find the point $Q$ on the plane in (a) that is closest to $P$.

(d) Find the area of the triangle with vertices at points $A$, $B$, and $C$.

*Answers:* (a) $-x + 3y + 2z = 3$ (b) $3\sqrt{14}/14 \approx 0.802$ (c) $Q = (17/14, 5/14, 11/7)$ (d) $\sqrt{126}/2$

4. Find a unit vector parallel to the $xz$-plane and perpendicular to vector

$$\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}.$$

*Answers:* $\pm\frac{1}{\sqrt{20}} \begin{bmatrix} -4 \\ 0 \\ 2 \end{bmatrix}$

5. Find all solutions of the following linear system. Identify all leading and free variables.

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + 2x_4 &= 2 \\ 2x_1 + 5x_2 - 8x_3 + 6x_4 &= 5 \\ 3x_1 + 4x_2 - 5x_3 + 2x_4 &= 4 \end{aligned}$$

*Answer:* The leading variables are $x_1$ and $x_2$. The free variables are $x_3$ and $x_4$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

6. Determine whether the lines

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + s \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$$

intersect and if they do, find their point of intersection.

*Answer:* The two lines intersect at point $(13, 8, 6)$, corresponding to $s = 4$ and $t = 2$.

7. Find the equation of the parabola passing through the three points:

$$(-1, 10), \ (1, 0), \ (2, 4).$$

Write your answer in the form $y = ax^2 + bx + c$.

*Answer:* $y = 3x^2 - 5x + 2$.

8. Find parametric equations for the line of intersection of the planes

$$3x + 2y - 4z = 6 \quad \text{and} \quad x - 3y - 2z = 4.$$

*Answer:* $x = \frac{26}{11} + \frac{16}{11}t, \quad y = -\frac{6}{11} - \frac{2}{11}t, \quad z = t$

9. Consider the following matrices.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -4 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 \\ 4 & 2 \\ 5 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 6 & 2 \\ 4 & 3 \end{bmatrix}$$

Evaluate: (a) $AB + 4C$ (b) $BA$ (c) $A + B^T$ (d) $C^{-1}$

*Answers:* (a) $AB + 4C = \begin{bmatrix} 47 & 16 \\ 13 & 5 \end{bmatrix}$ (b) $BA = \begin{bmatrix} 7 & 5 & 5 \\ 10 & 4 & 8 \\ 10 & 15 & 5 \end{bmatrix}$

(c) $A + B^T = \begin{bmatrix} 5 & 7 & 6 \\ 2 & -2 & 2 \end{bmatrix}$ (d) $C^{-1} = \begin{bmatrix} 3/10 & -1/5 \\ -2/5 & 3/5 \end{bmatrix}$

10. Consider the following matrix.

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & 1 \\ -3 & 3 & 4 \end{bmatrix}$$

(a) Find its inverse $A^{-1}$.

(b) Use your answer in (a) to solve the following system of linear equations.

$$\begin{cases} x - y - z = 2 \\ -y + z = 1 \\ -3x + 3y + 4z = 3 \end{cases}$$

*Answers:* (a) $A^{-1} = \begin{bmatrix} 7 & -1 & 2 \\ 3 & -1 & 1 \\ 3 & 0 & 1 \end{bmatrix}$ (b) $x = 19,\ y = 8,\ z = 9$

11. Solve for the matrix $X$ in the following matrix equation.

$$(A^{-1}X)^{-1} = (AB^{-1})^{-1}(AB^2).$$

Simplify your answer as much as possible.

*Answer:* $X = A(B^{-1})^3$

12. Find the matrix $A$ that satisfies the following.

$$\left(2A^{-1} + I_2\right)^T = \begin{bmatrix} 9 & 4 \\ 2 & 3 \end{bmatrix}$$

*Answer:* $A = \begin{bmatrix} 1/2 & -1/2 \\ -1 & 2 \end{bmatrix}$

13. Consider the matrix

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}$$

(a) Write $A^{-1}$ as a product of elementary matrices.

(b) Write $A$ as a product of elementary matrices.

*Answers:* Lots of answers are possible.

(a) $A^{-1} = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1/7 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$

14. Consider the following vectors.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix},\ \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Are $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ linearly dependent or linearly independent?

*Answer:* They are linearly dependent (LD). Observe that $\mathbf{v}_3 = 3\mathbf{v}_1 - \mathbf{v}_2$.

15. Determine if the three vectors in $\mathbb{R}^4$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 1 \end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix} 5 \\ 6 \\ -1 \\ 0 \end{bmatrix},\ \mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

are linearly dependent or linearly independent.

*Answer:* They are linearly independent (LI).

16. Consider the following three vectors in $\mathbb{R}^3$.

$$\mathbf{u} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{v_1} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Is vector $\mathbf{u}$ in span$(\mathbf{v_1}, \mathbf{v_2})$? If yes, express $\mathbf{u}$ as a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$.

*Answer:* The vector $\mathbf{u}$ is not in span$(\mathbf{v_1}, \mathbf{v_2})$.

17. Find an $LU$ factorization of the following matrix.

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 2 & 3 \\ -4 & -2 & 1 \end{bmatrix}$$

*Answer:* Lots of answers are possible.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & -3 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 9 \\ 0 & 0 & 16 \end{bmatrix}$$

18. Consider the following $3 \times 5$ matrix.

$$A = \begin{bmatrix} 2 & -4 & 0 & 2 & 1 \\ -1 & 2 & 1 & 2 & 3 \\ 1 & -2 & 1 & 4 & 4 \end{bmatrix}$$

(a) Find bases for the subspaces row$(A)$, col$(A)$, and null$(A)$.

(b) What are the values of rank$(A)$ and nullity$(A)$?

(c) What are all the possible values of rank$(A)$ and nullity$(A)$ for an arbitrary $3 \times 5$ matrix?

*Answers:*

(a) Basis of row$(A)$:

$$\begin{bmatrix} 1 & -2 & 0 & 1 & 1/2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 & 3 & 7/2 \end{bmatrix}$$

Basis of col$(A)$:

$$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Basis of null$(A)$:

$$\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1/2 \\ 0 \\ -7/2 \\ 0 \\ 1 \end{bmatrix}.$$

(b) $\text{rank}(A) = 2$ and $\text{nullity}(A) = 3$

(c) The possible values are the following.

| rank($A$) | null($A$) |
|---|---|
| 0 | 5 |
| 1 | 4 |
| 2 | 3 |
| 3 | 2 |

19. Consider the following four vectors in $\mathbb{R}^3$.

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(a) Show that $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ form a basis of $\mathbb{R}^3$.

(b) Find the coordinate vector $[\mathbf{u}]_{\mathcal{B}}$.

(c) Find $T(\mathbf{u})$ for the linear transformation $T \colon \mathbb{R}^3 \to \mathbb{R}^2$ given that

$$T(\mathbf{v}_1) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad T(\mathbf{v}_2) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad T(\mathbf{v}_3) = \begin{bmatrix} -1 \\ 7 \end{bmatrix}.$$

*Answers:*

(a) It is sufficient to show that $\text{RREF}([\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3]) = I_3$.

(b) $[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$

(c) $T(\mathbf{u}) = \begin{bmatrix} 19 \\ 10 \end{bmatrix}$

20. (a) Find the matrix associated to the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that reflects a vector through the line $y = x$, followed by a counterclockwise rotation of $60°$.

(b) Find the vector obtained if we apply the transformation described in part (a) to vector

$$\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

*Answers:*

(a) $\begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$

(b) $\begin{bmatrix} (-3\sqrt{3}+2)/2 \\ (3+2\sqrt{3})/2 \end{bmatrix} \approx \begin{bmatrix} -1.598 \\ 3.232 \end{bmatrix}$

21. Evaluate the following. Give your answer in the form $a + bi$.

(a) $\dfrac{1+3i}{2+5i}$ (b) $(1+2i)^{10}$

*Answers:* (a) $\frac{17}{29} + \frac{i}{29}$ (b) $237 - 3116i$

22. Find all complex numbers $z = a + bi$ that satisfy $z^3 = 27i$.

    *Answers:* $\frac{3\sqrt{3}}{2} + \frac{3}{2}i$, $\frac{-3\sqrt{3}}{2} + \frac{3}{2}i$, $-3i$.

23. Perform the following operations. Give your answers in phasor form.

    (a) $(3i)(2\angle 25^\circ)^4$

    (b) $\dfrac{(10\angle 112^\circ)(8\angle 228^\circ)}{(4\angle 48^\circ)(5\angle 72^\circ)}$

    (c) $3\angle 115.2^\circ + 5\angle 195.6^\circ$

    *Answers:* (a) $48\angle 190^\circ$ (b) $4\angle 220^\circ$ (c) $6.245\angle 167.3^\circ$

24. Use the cofactor method to find the inverse of the following matrix.

$$A = \begin{bmatrix} 2 & 4 & -1 \\ 0 & 3 & 1 \\ 6 & -2 & 5 \end{bmatrix}$$

*Answer:*

$$A^{-1} = \frac{1}{76}\begin{bmatrix} 17 & -18 & 7 \\ 6 & 16 & -2 \\ -18 & 28 & 6 \end{bmatrix}$$

25. Use Cramer's rule to solve the following system.

$$\begin{cases} 2x + 3y - 5z = 2 \\ 3x - y + 2z = 1 \\ 5x + 4y - 6z = 3 \end{cases}$$

*Answer:* $x = 3/5$, $y = -12/5$, $z = -8/5$

26. Find the eigenvalues and the corresponding eigenspaces of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Is the matrix diagonalizable?

*Answers:* The eigenvalues are: $\lambda_1 = 0$ (algebraic multiplicity 2) and $\lambda_2 = 3$. The eigenspaces are

$$E_{\lambda_1} = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$E_{\lambda_2} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

The geometric multiplicity of $\lambda_1$ is 2. Since the geometric and algebraic multiplicities of each eigenvalues are the same, we can conclude that the matrix is diagonalizable.

27. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

(a) Find an invertible matrix $P$ and a diagonal matrix $D$ such that

$$A = PDP^{-1}.$$

(b) Evaluate $A^{10}$.

*Answers:*

$$P = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} 1/3 & 1/3 \\ -2/3 & 1/3 \end{bmatrix}$$

$$A^{10} = \begin{bmatrix} 3\,255\,209 & 3\,255\,208 \\ 6\,510\,416 & 6\,510\,417 \end{bmatrix}$$

28. Find the eigenvalues of the rotation matrix

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

*Answers:* $e^{i\theta}$ and $e^{-i\theta}$

29. Let $W$ be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\mathbf{v}_1 = [1,4,5,6], \quad \mathbf{v}_2 = [3,-2,1,4], \quad \mathbf{v}_3 = [-1,0,-1,-2].$$

(a) Find a basis of $W^\perp$.

(b) Find an orthonormal basis of $W^\perp$.

*Answers:* (a) A basis of $W^\perp$ is $\{[-1,-1,1,0],\ [-2,-1,0,1]\}$.

(b) An orthonormal basis of $W^\perp$ is $\left\{\frac{1}{\sqrt{3}}[-1,-1,1,0],\ \frac{1}{\sqrt{3}}[-1,0,-1,1]\right\}$

30. Find $\text{proj}_W(\mathbf{v})$ if

$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} \right\} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 1 \end{bmatrix}.$$

*Answer:* $\text{proj}_W(\mathbf{v}) = \begin{bmatrix} 43/26 \\ 27/13 \\ 97/26 \\ 8/13 \end{bmatrix} \approx \begin{bmatrix} 1.6538 \\ 2.0769 \\ 3.7308 \\ 0.6154 \end{bmatrix}$

31. Orthogonally diagonalize the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

by finding an orthogonal matrix $Q$ and a diagonal matrix $D$ such that $A = QDQ^T$.

*Answer:*

$$Q = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

32. Find the least squares solution of the following system.

$$\begin{cases} x + y = -3 \\ 2x + 3y = -1 \\ -3x + 2y = 2 \end{cases}$$

*Answer:* $x = -152/195$ and $y = -17/195$

33. Find the least squares straight line fit to the four points: $(0, 1)$, $(1, 3)$, $(2, 4)$, and $(3, 4)$.
*Answer:* $y = 1.5 + x$