

Linear Transformations

Definition: A transformation (or function or mapping) $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called a **linear transformation** if

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all \mathbf{u} and \mathbf{v} in \mathbb{R}^n , and
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all \mathbf{u} in \mathbb{R}^n and for all scalars c .

Theorem: Let A be an $m \times n$ matrix. Then the transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by

$$T(\mathbf{x}) = A\mathbf{x},$$

for all \mathbf{x} in \mathbb{R}^n , is a linear transformation.

Proof: Let \mathbf{u} and \mathbf{v} be in \mathbb{R}^n and let c be a scalar. Then

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v}), \text{ and} \\ T(c\mathbf{u}) &= A(c\mathbf{u}) = c(A\mathbf{u}) = cT(\mathbf{u}). \end{aligned}$$

Therefore, T is a linear transformation.

Theorem: Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then there is an $m \times n$ matrix A (called the **standard matrix of the linear transformation**) such that

$$T(\mathbf{x}) = A\mathbf{x},$$

for all \mathbf{x} in \mathbb{R}^n . In particular, if $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ represent the standard basis vectors in \mathbb{R}^n , then the columns of A are $T(\mathbf{e}_1), T(\mathbf{e}_2), \dots, T(\mathbf{e}_n)$.