Gram-Schmidt Process

The Gram-Schmidt Process: Let $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ be a basis for a subspace W of $\mathbb{R}^n$ and define the following:

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1, & W_1 &= \operatorname{span}(\mathbf{x}_1)\\
\mathbf{v}_2 &= \mathbf{x}_2 - \operatorname{proj}_{\mathbf{v}_1}(\mathbf{x}_2), & W_2 &= \operatorname{span}(\mathbf{x}_1, \mathbf{x}_2)\\
\mathbf{v}_3 &= \mathbf{x}_3 - \operatorname{proj}_{\mathbf{v}_1}(\mathbf{x}_3) - \operatorname{proj}_{\mathbf{v}_2}(\mathbf{x}_3), & W_3 &= \operatorname{span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)\\
&\vdots\\
\mathbf{v}_k &= \mathbf{x}_k - \operatorname{proj}_{\mathbf{v}_1}(\mathbf{x}_k) - \operatorname{proj}_{\mathbf{v}_2}(\mathbf{x}_k) - \dots - \operatorname{proj}_{\mathbf{v}_{k-1}}(\mathbf{x}_k), & W_k &= \operatorname{span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_k)
\end{aligned}$$

Then for each $i = 1, 2, \dots, k$, $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_i\}$ is an orthogonal basis for W_i. In particular, $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is an orthogonal basis for W.