Converting Equations of Lines and Planes

The equations of lines in $\mathbb{R}^2$ or $\mathbb{R}^3$ and planes in $\mathbb{R}^3$ can be written in any of four forms: normal, general, vector or parametric, as summarized below. We look at how such equations can be converted between these different forms.

Equations of Lines in $\mathbb{R}^2$

Normal Form	$\mathbf{n}\cdot\mathbf{x} = \mathbf{n}\cdot\mathbf{p}$ or $\mathbf{n}\cdot(\mathbf{x}-\mathbf{p}) = 0$
General Form	$ax + by = c$
Vector Form	$\mathbf{x} = \mathbf{p} + t\mathbf{d}$
Parametric Form	$\begin{cases} x = p_1 + td_1 \\ y = p_2 + td_2 \end{cases}$

Equations of Lines in $\mathbb{R}^3$

Normal Form	$\begin{cases} \mathbf{n}_1\cdot\mathbf{x} = \mathbf{n}_1\cdot\mathbf{p}_1 \\ \mathbf{n}_2\cdot\mathbf{x} = \mathbf{n}_2\cdot\mathbf{p}_2 \end{cases}$ or $\begin{cases} \mathbf{n}_1\cdot(\mathbf{x}-\mathbf{p}_1) = 0 \\ \mathbf{n}_2\cdot(\mathbf{x}-\mathbf{p}_2) = 0 \end{cases}$
General Form	$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \end{cases}$
Vector Form	$\mathbf{x} = \mathbf{p} + t\mathbf{d}$
Parametric Form	$\begin{cases} x = p_1 + td_1 \\ y = p_2 + td_2 \\ z = p_3 + td_3 \end{cases}$

Equations of Planes in $\mathbb{R}^3$

Normal Form	$\mathbf{n}\cdot\mathbf{x} = \mathbf{n}\cdot\mathbf{p}$ or $\mathbf{n}\cdot(\mathbf{x}-\mathbf{p}) = 0$
General Form	$ax + by + cz = d$
Vector Form	$\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$
Parametric Form	$\begin{cases} x = p_1 + su_1 + tv_1 \\ y = p_2 + su_2 + tv_2 \\ z = p_3 + su_3 + tv_3 \end{cases}$

1. **Convert lines in $\mathbb{R}^2$ or $\mathbb{R}^3$ and planes in $\mathbb{R}^3$ between vector and parametric forms**

This is the easiest type of conversion since vector and parametric forms are really just two slightly different ways of writing the same thing, either as separate parametric equations, one for each variable, or as a single vector equation. Consider the following examples.

Vector Form	$\longleftrightarrow$	**Parametric Form**
$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \end{bmatrix}$		$\begin{cases} x = 3 - t \\ y = 4 + 2t \end{cases}$
$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ -4 \end{bmatrix} + t \begin{bmatrix} 6 \\ 5 \\ -4 \end{bmatrix}$		$\begin{cases} x = 7 + 6t \\ y = 5t \\ z = -4 - 4t \end{cases}$
$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + t \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$		$\begin{cases} x = 1 + 4s + 7t \\ y = 2 + 5s + 8t \\ z = 3 + 6s + 9t \end{cases}$

Note that the constant terms in the parametric equations are associated with vector $\mathbf{p}$, which corresponds to a point on the line or plane, while the coefficients of the parameter t (or s and t in the case of a plane) form the direction vector $\mathbf{d}$ of the line (or direction vectors $\mathbf{u}$ and $\mathbf{v}$ of the plane).

2. **Convert lines in $\mathbb{R}^2$ or $\mathbb{R}^3$ and planes in $\mathbb{R}^3$ between normal and general forms**

This is the next simplest type of conversion. To go from normal form to general form, we just expand and simplify the dot products. For example,

Normal Form	$\longrightarrow$	**General Form**
$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -6 \end{bmatrix}$		$2x + 3y = -16$
$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 0 \\ -1 \end{bmatrix}$		$x + 2y + 3z = 4.$

To convert from general to normal form, let $\mathbf{n}$ be the vector whose components are the coefficients of x and y in $\mathbb{R}^2$ or x, y, and z in $\mathbb{R}^3$. Then find any point P on the line or plane to form $\mathbf{p}$. There are lots of points to choose from. For example,

General Form $\longrightarrow$ **Normal Form**

$2x + 3y = -16$

Let $\mathbf{n} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Pick a point on the line. For example, if $y = 0$, then $x = -8$, and so let $\mathbf{p} = \begin{bmatrix} -8 \\ 0 \end{bmatrix}$. Then we get

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -8 \\ 0 \end{bmatrix}.$$

Note that if we used the point $(1, -6)$, so that $\mathbf{p} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$, we would get the same normal form as before.

$7x - 2y + 3z = 4$

Let $\mathbf{n} = \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix}$. Pick a point on the plane. For example, if $x = z = 0$, then $y = -2$, and so let $\mathbf{p} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$. Then we get

$$\begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}.$$

So, how do we convert between vector/parametric and normal/general forms? Let's begin by considering planes in $\mathbb{R}^3$.

3. **Convert planes in $\mathbb{R}^3$ from normal/general to vector/parametric form**

Given a plane $ax + by + cz = d$, we first find three points A, B and C on the plane and then compute two direction vectors (e.g. $\mathbf{u} = \overrightarrow{AB}$ and $\mathbf{v} = \overrightarrow{AC}$), making sure they are not parallel. If they are parallel, it means the chosen points happen to lie on the same line (i.e. they are collinear). If that's the case, then pick different points. Lastly, use one of the points to form $\mathbf{p}$.

Normal/General Form $\longrightarrow$ **Vector/Parametric Form**

$7x - 2y + 3z = 4$

Find three points (not collinear) on the plane, for example $A = (0, -2, 0)$, $B = (1, 0, -1)$ and $C = (1, 3, 1)$. If we let $\mathbf{u} = \overrightarrow{AB} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \overrightarrow{AC} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$, and we let $\mathbf{p} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$ correspond to point A, then we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}.$$

4. **Convert planes in $\mathbb{R}^3$ from vector/parametric to normal/general form**

Given a plane $\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$, we can find a normal vector $\mathbf{n}$ by calculating the cross product $\mathbf{n} = \mathbf{u} \times \mathbf{v}$. Using the same $\mathbf{p}$, we can then get the normal form $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$.

Vector/Parametric Form $\longrightarrow$ **Normal/General Form**

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$$

Let $\mathbf{p} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$, and compute $\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix}$. Then we get the normal form

$$\begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix},$$

or $7x - 2y + 3z = 4$ if expanded into general form.

5. **Convert lines in $\mathbb{R}^2$ between normal/general and vector/parametric form**

We note that a nonzero vector $\mathbf{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ is a normal vector for a line in $\mathbb{R}^2$ if and only if $\mathbf{d} = \begin{bmatrix} b \\ -a \end{bmatrix}$ is a direction vector, since $\mathbf{n} \cdot \mathbf{d} = ab - ba = 0$, and so $\mathbf{n}$ and $\mathbf{d}$ are orthogonal. Also, the same vector $\mathbf{p}$ can be used for both normal and vector forms.

Normal/General Form $\longrightarrow$ **Vector/Parametric Form**

$$\begin{bmatrix}2\\3\end{bmatrix} \cdot \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix} \cdot \begin{bmatrix}1\\-6\end{bmatrix}$$

Since $\mathbf{n} = \begin{bmatrix}2\\3\end{bmatrix}$, then $\mathbf{d} = \begin{bmatrix}3\\-2\end{bmatrix}$ is a direction vector. Letting $\mathbf{p} = \begin{bmatrix}1\\-6\end{bmatrix}$ gives us

$$\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\-6\end{bmatrix} + t\begin{bmatrix}3\\-2\end{bmatrix}.$$

Vector/Parametric Form $\longrightarrow$ **Normal/General Form**

$$\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\-6\end{bmatrix} + t\begin{bmatrix}3\\-2\end{bmatrix}$$

Since $\mathbf{d} = \begin{bmatrix}3\\-2\end{bmatrix}$, then $\mathbf{n} = \begin{bmatrix}2\\3\end{bmatrix}$ is a normal vector. Letting $\mathbf{p} = \begin{bmatrix}1\\-6\end{bmatrix}$ gives us

$$\begin{bmatrix}2\\3\end{bmatrix} \cdot \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix} \cdot \begin{bmatrix}1\\-6\end{bmatrix},$$

or $2x + 3y = -16$ if expanded into general form.

6. **Convert lines in $\mathbb{R}^3$ from vector/parametric to normal/general form**

Given the vector form $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, we can let $\mathbf{p}_1 = \mathbf{p}$ and $\mathbf{p}_2 = \mathbf{p}$, and then find two nonparallel vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ that are orthogonal to $\mathbf{d}$; there are lots to choose from. Note that the normal form for the equation of a line in $\mathbb{R}^3$ consists of a pair of equations representing intersecting planes.

Vector/Parametric Form $\longrightarrow$ **Normal/General Form**

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 5 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Since $\mathbf{p} = \begin{bmatrix} 7 \\ -2 \\ 5 \end{bmatrix}$, then let $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}$. Since $\mathbf{d} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then we can find two nonparallel vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ such that $\mathbf{n}_1 \cdot \mathbf{d} = 0$ and $\mathbf{n}_2 \cdot \mathbf{d} = 0$. For example, pick $\mathbf{n}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{n}_2 = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}$ by using the same strategy we used for lines in $\mathbb{R}^2$ while keeping one component zero. Then we get

$$\begin{cases} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ -2 \\ 5 \end{bmatrix} \\ \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ -2 \\ 5 \end{bmatrix} \end{cases},$$

or

$$\begin{cases} -2x + y = -16 \\ -3y + 2z = 16 \end{cases}$$

if expanded into general form.

7. **Convert lines in $\mathbb{R}^3$ from normal/general to vector/parametric form**

The general form of a line in $\mathbb{R}^3$ consists of a pair of equations of planes,

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2. \end{cases}$$

While we could find a direction vector $\mathbf{d}$ for the line by taking the cross product of the normal vectors $\mathbf{n}_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$ and $\mathbf{n}_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$ since $\mathbf{d}$ must be orthogonal to both $\mathbf{n}_1$ and $\mathbf{n}_2$, that still leaves us with having to find a point on the line from which to get $\mathbf{p}$. Instead, we solve the system of equations and introduce a parameter representing a free variable. This will allow us to find $\mathbf{p}$ and $\mathbf{d}$ without the need for cross products.

Normal/General Form $\longrightarrow$ **Vector/Parametric Form**

$$\begin{cases} -2x + y \quad = -16 \\ \quad - 3y + 2z = 16 \end{cases}$$

We can solve the system by row reducing its augmented matrix:

$$\left[\begin{array}{ccc|c} -2 & 1 & 0 & -16 \\ 0 & -3 & 2 & 16 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & -1/3 & 16/3 \\ 0 & 1 & -2/3 & -16/3 \end{array}\right]$$

Letting the free variable z be t, we get $x = \frac{1}{3}t + \frac{16}{3}$ and $y = \frac{2}{3}t - \frac{16}{3}$, and so the solution of the system, and therefore the equation of the line in vector form, is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16/3 \\ -16/3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1/3 \\ 2/3 \\ 1 \end{bmatrix},$$

or by scaling the direction vector and redefining t, we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16/3 \\ -16/3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Note that $(7, -2, 5)$ and $(16/3, -16/3, 0)$ are both points on the line, corresponding to $t = 5/3$ and $t = 0$ respectively.