CAMOSUN

Name: SOLUTIONS

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MATH 251
Assignment 7

1. (4 marks) Suppose

$$A = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 2 & -4 & 2 & 4 \\ -1 & 2 & 1 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and $W = \text{col}(A)$ is the column space of A.

(a) Find an orthogonal basis for W.

(b) Find $\text{proj}_W(\mathbf{v})$ and $\text{perp}_W(\mathbf{v})$, where $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$ is a basis for W, but it's not orthogonal.

So we apply Gram-Schmidt. Let $\vec{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

$$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{v}_1}(\vec{x}_2) = \vec{x}_2 - \left(\frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \right) \vec{v}_1 = \vec{x}_2 - \frac{6}{6}\vec{v}_1 = \vec{x}_2 - \vec{v}_1$$

$$= \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \xrightarrow{\text{scale}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ An orthogonal basis for W is $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

b) $$\text{proj}_W(\vec{v}) = \text{proj}_{\vec{v}_1}(\vec{v}) + \text{proj}_{\vec{v}_2}(\vec{v}) = \left(\frac{\vec{v} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \right)\vec{v}_1 + \left(\frac{\vec{v} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \right)\vec{v}_2$$

$$= \frac{2}{6}\vec{v}_1 + \frac{4}{2}\vec{v}_2 = \frac{1}{3}\vec{v}_1 + 2\vec{v}_2 = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 2/3 \\ 5/3 \end{bmatrix} \text{ or } \frac{1}{3}\begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}$$

$$\text{perp}_W(\vec{v}) = \vec{v} - \text{proj}_W(\vec{v}) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 7/3 \\ 2/3 \\ 5/3 \end{bmatrix} = \begin{bmatrix} -4/3 \\ 4/3 \\ 4/3 \end{bmatrix} \text{ or } \frac{4}{3}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

2. (5 marks) Find a QR factorization of $A = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\ \vec{x}_1 \ \ \vec{x}_2 \ \ \vec{x}_3 \end{array}$.

Let $\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$\vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{v}_1}(\vec{x}_2) = \vec{x}_2 - \left(\frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 = \vec{x}_2 - \frac{3}{4}\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} \xrightarrow{\text{scale}} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{v}_3 = \vec{x}_3 - \text{proj}_{\vec{v}_1}(\vec{x}_3) - \text{proj}_{\vec{v}_2}(\vec{x}_3) = \vec{x}_3 - \left(\frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 - \left(\frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\right)\vec{v}_2$$

$$= \vec{x}_3 - \frac{2}{4}\vec{v}_1 - \frac{2}{12}\vec{v}_2 = \vec{x}_3 - \frac{1}{2}\vec{v}_1 - \frac{1}{6}\vec{v}_2$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix} \xrightarrow{\text{scale}} \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

$$Q = \left[\frac{1}{\|\vec{v}_1\|}\vec{v}_1 \,\middle|\, \frac{1}{\|\vec{v}_2\|}\vec{v}_2 \,\middle|\, \frac{1}{\|\vec{v}_3\|}\vec{v}_3\right] = \left[\frac{1}{2}\vec{v}_1 \,\middle|\, \frac{1}{2\sqrt{3}}\vec{v}_2 \,\middle|\, \frac{1}{\sqrt{6}}\vec{v}_3\right] = \begin{bmatrix} \frac{1}{2} & \frac{-3}{2\sqrt{3}} & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{-3}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3/2 & 1 \\ 0 & 3/2\sqrt{3} & 1/\sqrt{3} \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix}$$

$$\therefore A = QR = \begin{bmatrix} \frac{1}{2} & \frac{-3}{2\sqrt{3}} & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 2 & 3/2 & 1 \\ 0 & \frac{3}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$$

3. (4 marks) Orthogonally diagonalize the following matrix, whose characteristic polynomial is $-(\lambda-2)(\lambda-8)^2$, by finding an orthogonal matrix Q and a diagonal matrix D such that $Q^TAQ = D$.

$$A = \begin{bmatrix} 5 & 0 & 3 \\ 0 & 8 & 0 \\ 3 & 0 & 5 \end{bmatrix}$$

$\underline{\lambda_1 = 2}$ $A - \lambda_1 I = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 6 & 0 \\ 3 & 0 & 3 \end{bmatrix}$ Solve $(A-\lambda_1 I)\vec{x} = \vec{0}$

$$\left[\begin{array}{ccc|c} 3 & 0 & 3 & 0 \\ 0 & 6 & 0 & 0 \\ 3 & 0 & 3 & 0 \end{array}\right] \xrightarrow[\substack{\frac{1}{6}R_2 \\ R_3 - R_1}]{\frac{1}{3}R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{aligned} x &= -t \\ y &= 0 \\ z &= t \end{aligned} \quad \vec{x} = t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Basis for E_{λ_1} is $\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$

$\underline{\lambda_2 = 8}$ $A - \lambda_2 I = \begin{bmatrix} -3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & -3 \end{bmatrix}$ Solve $(A-\lambda_2 I)\vec{x} = \vec{0}$

$$\left[\begin{array}{ccc|c} -3 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & -3 & 0 \end{array}\right] \xrightarrow[R_3 + R_1]{-\frac{1}{3}R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{aligned} x &= t \\ y &= s \\ z &= t \end{aligned} \quad \vec{x} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Basis for E_{λ_2} is $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$ Already orthogonal, so no need for Gram-Schmidt.

Let $\vec{v_1} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Then $Q = \left[\frac{1}{\|\vec{v_1}\|}\vec{v_1} \,\middle|\, \frac{1}{\|\vec{v_2}\|}\vec{v_2} \,\middle|\, \frac{1}{\|\vec{v_3}\|}\vec{v_3}\right] = \left[\frac{1}{\sqrt{2}}\vec{v_1} \,\middle|\, \vec{v_2} \,\middle|\, \frac{1}{\sqrt{2}}\vec{v_3}\right]$

$$= \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \quad \text{and } D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}.$$

Then $Q^TAQ = D$.

4. (5 marks) Suppose A is a 3×3 symmetric matrix having eigenvalues $\lambda_1 = -9$ and $\lambda_2 = 9$ and corresponding eigenspaces $E_{\lambda_1} = \text{span}\left(\begin{bmatrix}-2\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\1\end{bmatrix}\right)$ and $E_{\lambda_2} = \text{span}\left(\begin{bmatrix}1\\2\\2\end{bmatrix}\right)$.

   (a) Orthogonally diagonalize A by finding an orthogonal matrix Q and a diagonal matrix D such that $Q^TAQ = D$.

   (b) Find A.

a) Basis for E_{λ_1} is not orthogonal, so we apply Gram-Schmidt.

Let $\vec{x}_1 = \begin{bmatrix}-2\\1\\0\end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix}-2\\0\\1\end{bmatrix}$.

$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix}-2\\1\\0\end{bmatrix}$

$\vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{v}_1}(\vec{x}_2) = \vec{x}_2 - \left(\frac{\vec{x}_2\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1 = \vec{x}_2 - \frac{4}{5}\vec{v}_1$

$= \begin{bmatrix}-2\\0\\1\end{bmatrix} - \frac{4}{5}\begin{bmatrix}-2\\1\\0\end{bmatrix} = \begin{bmatrix}-2/5\\-4/5\\1\end{bmatrix} \xrightarrow{\text{scale}} \begin{bmatrix}-2\\-4\\5\end{bmatrix}$

Let $\vec{v}_3 = \begin{bmatrix}1\\2\\2\end{bmatrix}$ from E_{λ_2}.

Then $Q = \left[\frac{1}{\|\vec{v}_1\|}\vec{v}_1 \,\middle|\, \frac{1}{\|\vec{v}_2\|}\vec{v}_2 \,\middle|\, \frac{1}{\|\vec{v}_3\|}\vec{v}_3\right] = \left[\frac{1}{\sqrt{5}}\vec{v}_1 \,\middle|\, \frac{1}{3\sqrt{5}}\vec{v}_2 \,\middle|\, \frac{1}{3}\vec{v}_3\right]$

$= \begin{bmatrix}-\frac{2}{\sqrt{5}} & \frac{-2}{3\sqrt{5}} & \frac{1}{3}\\ \frac{1}{\sqrt{5}} & -\frac{4}{3\sqrt{5}} & \frac{2}{3}\\ 0 & \frac{5}{3\sqrt{5}} & \frac{2}{3}\end{bmatrix}$ and $D = \begin{bmatrix}-9 & 0 & 0\\ 0 & -9 & 0\\ 0 & 0 & 9\end{bmatrix}$

Then $Q^TAQ = D$.

b) $A = QDQ^T = \begin{bmatrix}-\frac{2}{\sqrt{5}} & \frac{-2}{3\sqrt{5}} & \frac{1}{3}\\ \frac{1}{\sqrt{5}} & \frac{-4}{3\sqrt{5}} & \frac{2}{3}\\ 0 & \frac{5}{3\sqrt{5}} & \frac{2}{3}\end{bmatrix}\begin{bmatrix}-9 & 0 & 0\\ 0 & -9 & 0\\ 0 & 0 & 9\end{bmatrix}\begin{bmatrix}-\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0\\ -\frac{2}{3\sqrt{5}} & \frac{-4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}}\\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3}\end{bmatrix}$

$= \begin{bmatrix}\frac{18}{\sqrt{5}} & \frac{6}{\sqrt{5}} & 3\\ -\frac{9}{\sqrt{5}} & \frac{12}{\sqrt{5}} & 6\\ 0 & -\frac{15}{\sqrt{5}} & 6\end{bmatrix}\begin{bmatrix}-\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0\\ -\frac{2}{3\sqrt{5}} & \frac{-4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}}\\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3}\end{bmatrix} = \begin{bmatrix}-7 & 4 & 4\\ 4 & -1 & 8\\ 4 & 8 & -1\end{bmatrix}$

5. (3 marks) Find the least squares solution of the following inconsistent linear system.

$$\begin{cases} x + \ y - 2z = 21 \\ x + 2y + \ z = 7 \\ 2x - \ y + \ z = 98 \\ x - \ y - \ z = 91 \end{cases}$$

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & 1 \\ 2 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 21 \\ 7 \\ 98 \\ 91 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & -1 \\ -2 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & 1 \\ 2 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} = 7I$$

$$\therefore (A^T A)^{-1} = \frac{1}{7} I$$

$$A^T \vec{b} = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & -1 \\ -2 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 21 \\ 7 \\ 98 \\ 91 \end{bmatrix} = \begin{bmatrix} 315 \\ -154 \\ -28 \end{bmatrix}$$

$$\vec{x}_{LS} = (A^T A)^{-1} (A^T \vec{b}) = \left(\frac{1}{7} I\right)(A^T \vec{b}) = \frac{1}{7} \begin{bmatrix} 315 \\ -154 \\ -28 \end{bmatrix} = \begin{bmatrix} 45 \\ -22 \\ -4 \end{bmatrix}$$

6. (4 marks) Find the least squares approximating line for the points

$$(-4, 3), (-2, -1), (-1, -2), (0, 1), (2, 4)$$

and compute the corresponding least squares error, rounded to two decimal places:

Want $y = a + bx$, where

$(-4,3)$ $a - 4b = 3$
$(-2,-1)$ $a - 2b = -1$
$(-1,-2)$ $a - b = -2$
$(0,1)$ $a = 1$
$(2,4)$ $a + 2b = 4$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & -4 \\ 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 3 \\ -1 \\ -2 \\ 1 \\ 4 \end{bmatrix}}_{\vec{b}}$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -4 & -2 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ -5 & 25 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{100} \begin{bmatrix} 25 & 5 \\ 5 & 5 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -4 & -2 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ -2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$$\vec{x}_{LS} = (A^TA)^{-1}(A^T\vec{b}) = \frac{1}{20} \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 25 \\ 5 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/4 \\ 1/4 \end{bmatrix}$$

$$\therefore \quad y = \frac{5}{4} + \frac{1}{4}x \quad \text{(best-fit line)}$$

$$A\vec{x}_{LS} = \begin{bmatrix} 1 & -4 \\ 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 3/4 \\ 1 \\ 5/4 \\ 7/4 \end{bmatrix}; \quad \vec{e} = \vec{b} - A\vec{x}_{LS} = \begin{bmatrix} 3 \\ -1 \\ -2 \\ 1 \\ 4 \end{bmatrix} - \begin{bmatrix} 1/4 \\ 3/4 \\ 1 \\ 5/4 \\ 7/4 \end{bmatrix} = \begin{bmatrix} 11/4 \\ -7/4 \\ -3 \\ -1/4 \\ 9/4 \end{bmatrix}$$

$$\|\vec{e}\| = \frac{1}{4}\sqrt{11^2 + (-7)^2 + (-12)^2 + (-1)^2 + 9^2} = \frac{1}{4}\sqrt{396} = \frac{3}{2}\sqrt{11} \approx 4.97 \quad \text{(error)}$$