CAMOSUN

Name: SOLUTIONS

Mark: $\overline{}$ **25**

MATH 251
Assignment 6

1. (9 marks) Consider the matrix

$$A = \begin{bmatrix} -3 & -4 & 4 \\ 4 & 5 & -4 \\ 2 & 2 & -1 \end{bmatrix}.$$

   (a) Find the eigenvalues of A.

   (b) Is A invertible? Briefly explain.

a) $\det(A-\lambda I) = \begin{vmatrix} -3-\lambda & -4 & 4 \\ 4 & 5-\lambda & -4 \\ 2 & 2 & -1-\lambda \end{vmatrix}$

$= (-3-\lambda)\begin{vmatrix} 5-\lambda & -4 \\ 2 & -1-\lambda \end{vmatrix} - (-4)\begin{vmatrix} 4 & -4 \\ 2 & -1-\lambda \end{vmatrix} + 4\begin{vmatrix} 4 & 5-\lambda \\ 2 & 2 \end{vmatrix}$

$= (-3-\lambda)\left[(5-\lambda)(-1-\lambda)+8\right] + 4(-4-4\lambda+8) + 4(8-10+2\lambda)$

$= (-3-\lambda)(\lambda^2-4\lambda+3) + 4(-4\lambda+4) + 4(2\lambda-2)$

$= -\lambda^3+\lambda^2+9\lambda-9-16\lambda+16+8\lambda-8 \quad = -\lambda^3+\lambda^2+\lambda-1$

$= -\lambda^2(\lambda-1)+(\lambda-1) \quad = (-\lambda^2+1)(\lambda-1) \quad = -(\lambda+1)(\lambda-1)(\lambda-1)$

$= -(\lambda+1)(\lambda-1)^2 = 0 \Rightarrow \lambda_1 = -1$ or $\lambda_2 = 1$

(alg. mult. 1) (alg. mult. 2)

b) Yes. A^{-1} exists by the Fundamental Theorem of Invertible Matrices since 0 is not an eigenvalue of A.

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(c) Find the eigenspaces corresponding to each eigenvalue of A.

$\lambda_1 = -1$ $A - \lambda_1 I = \begin{bmatrix} -2 & -4 & 4 \\ 4 & 6 & -4 \\ 2 & 2 & 0 \end{bmatrix}$ Solve $(A - \lambda_1 I)\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccc|c} -2 & -4 & 4 & 0 \\ 4 & 6 & -4 & 0 \\ 2 & 2 & 0 & 0 \end{array}\right] \xrightarrow[\substack{R_2 + 2R_1 \\ R_3 + R_1}]{-\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & -2 & 4 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{-\frac{1}{2}R_2 \\ R_3 - R_2}]{R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$z = t$ (free)

$x = -2t$ and $y = 2t$

$\therefore \vec{x} = t\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$ and so $E_{\lambda_1} = \text{span}\left(\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}\right)$

(geo. mult. 1)

$\lambda_2 = 1$ $A - \lambda_2 I = \begin{bmatrix} -4 & -4 & 4 \\ 4 & 4 & -4 \\ 2 & 2 & -2 \end{bmatrix}$ Solve $(A - \lambda_2 I)\vec{x} = \vec{0}$

$$\left[\begin{array}{ccc|c} -4 & -4 & 4 & 0 \\ 4 & 4 & -4 & 0 \\ 2 & 2 & -2 & 0 \end{array}\right] \xrightarrow[\substack{R_2 + R_1 \\ R_3 + \frac{1}{2}R_1}]{-\frac{1}{4}R_1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$y = s$, $z = t$ (free)

$x = -s + t$

$\therefore \vec{x} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and so $E_{\lambda_2} = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$

(geo. mult. 2)

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(d) Is A diagonalizable? Briefly explain. If A is diagonalizable, then find an invertible matrix P and a diagonal matrix D so that $P^{-1}AP = D$.

(e) Evaluate A^{100}.

(d) Yes. A is diagonalizable since alg. mult. = geo. mult. for all eigenvalues

Let $P = \begin{bmatrix} -2 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then $P^{-1}AP = D$

(e) $A^{100} = PD^{100}P^{-1} = P\begin{bmatrix} (-1)^{100} & 0 & 0 \\ 0 & 1^{100} & 0 \\ 0 & 0 & 1^{100} \end{bmatrix}P^{-1} = P\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}P^{-1}$

$= PIP^{-1} = PP^{-1} = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(Note: It's not necessary to calculate P^{-1}!)

2. (4 marks) Find the eigenvalues and corresponding eigenspaces of $A = \begin{bmatrix} 3 & -5 \\ 1 & 7 \end{bmatrix}$.

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -5 \\ 1 & 7-\lambda \end{vmatrix} = (3-\lambda)(7-\lambda)+5 = \lambda^2 - 10\lambda + 26$$

$$= \lambda^2 - 10\lambda + 25 + 1 = (\lambda-5)^2 + 1 = 0 \Rightarrow (\lambda-5)^2 = -1$$

$$\Rightarrow \lambda - 5 = \pm i \Rightarrow \lambda = 5 \pm i$$

$\underline{\lambda_1 = 5+i}$ $\quad A - \lambda_1 I = \begin{bmatrix} -2-i & -5 \\ 1 & 2-i \end{bmatrix}$ $\quad$ Solve $(A-\lambda_1 I)\vec{x} = \vec{0}$

$$\left[\begin{array}{cc|c} -2-i & -5 & 0 \\ 1 & 2-i & 0 \end{array}\right] \xrightarrow{(-2+i)R_1} \left[\begin{array}{cc|c} 5 & 10-5i & 0 \\ 1 & 2-i & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{5}R_1} \left[\begin{array}{cc|c} 1 & 2-i & 0 \\ 1 & 2-i & 0 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cc|c} 1 & 2-i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$y = t$ (free)
$x = (-2+i)t$

$\therefore \vec{x} = t\begin{bmatrix} -2+i \\ 1 \end{bmatrix}$ and so $E_{\lambda_1} = \text{span}\left(\begin{bmatrix} -2+i \\ 1 \end{bmatrix}\right)$

$\underline{\lambda_2 = 5-i}$ $\quad \therefore E_{\lambda_2} = \text{span}\left(\begin{bmatrix} -2-i \\ 1 \end{bmatrix}\right)$

(complex conjugate)

3. (2 marks) For what value(s) of k (if any) is the set

$$\mathcal{B} = \left\{ \begin{bmatrix} k \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -k \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ k \\ 2 \end{bmatrix} \right\}$$

an orthogonal basis of $\mathbb{R}^3$?

$$\begin{bmatrix} K \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -K \\ 1 \end{bmatrix} = 0 \checkmark \qquad \begin{bmatrix} K \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ K \\ 2 \end{bmatrix} = 0 \checkmark$$

$$\begin{bmatrix} 1 \\ -K \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ K \\ 2 \end{bmatrix} = -K^2 + 1 \leftarrow \text{must be } 0$$

$$\therefore K = \pm 1$$

4. (2 marks) Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix},$$

and note that $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ forms an orthogonal basis of $\mathbb{R}^4$. Express $\mathbf{u}$ as a linear combination of these basis vectors. Give the coordinate vector $[\mathbf{u}]_{\mathcal{B}}$ of $\mathbf{u}$ with respect to $\mathcal{B}$.

$$\vec{u} = \left(\frac{\vec{u}\cdot\vec{v_1}}{\vec{v_1}\cdot\vec{v_1}}\right)\vec{v_1} + \left(\frac{\vec{u}\cdot\vec{v_2}}{\vec{v_2}\cdot\vec{v_2}}\right)\vec{v_2} + \left(\frac{\vec{u}\cdot\vec{v_3}}{\vec{v_3}\cdot\vec{v_3}}\right)\vec{v_3} + \left(\frac{\vec{u}\cdot\vec{v_4}}{\vec{v_4}\cdot\vec{v_4}}\right)\vec{v_4}$$

$$= \frac{8}{4}\vec{v_1} + \frac{6}{4}\vec{v_2} + \frac{4}{4}\vec{v_3} + \frac{2}{4}\vec{v_4}$$

$$= 2\vec{v_1} + \frac{3}{2}\vec{v_2} + \vec{v_3} + \frac{1}{2}\vec{v_4}$$

$$[\vec{u}]_B = \begin{bmatrix} 2 \\ 3/2 \\ 1 \\ 1/2 \end{bmatrix}$$

5. (1 mark) Find the inverse of the following orthogonal matrix.

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = A^T = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

6. (3 marks) Find a basis for $W^\perp$ if $W = \text{span}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ and

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 3 \\ -9 \\ 6 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -1 \\ -2 \\ 6 \\ -4 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 2 \\ 7 \\ -21 \\ 14 \end{bmatrix}.$$

Let $A = [\vec{w}_1 | \vec{w}_2 | \vec{w}_3]$. Then $W = \text{col}(A)$ and so $W^\perp = \text{null}(A^T)$

$$[A^T | \vec{0}] = \left[\begin{array}{cccc|c} 1 & 3 & -9 & 6 & 0 \\ -1 & -2 & 6 & -4 & 0 \\ 2 & 7 & -21 & 14 & 0 \end{array}\right] \xrightarrow{\substack{R_2+R_1 \\ R_3-2R_1}} \left[\begin{array}{cccc|c} 1 & 3 & -9 & 6 & 0 \\ 0 & 1 & -3 & 2 & 0 \\ 0 & 1 & -3 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{\substack{R_1-3R_2 \\ R_3-R_2}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = s$, $x_4 = t$ (free)
$x_1 = 0$, $x_2 = 3s - 2t$

$$\therefore \vec{x} = s\begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ A basis for $W^\perp$ is $\left\{ \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

7. (4 marks) Consider the vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -2 \\ 2 \\ 3 \\ 2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} -7 \\ -8 \\ 7 \\ 1 \end{bmatrix}.$$

Suppose $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis for a subspace W of $\mathbb{R}^4$. Note that $\mathcal{B}$ is an orthogonal set.

(a) Find the orthogonal projection of $\mathbf{v}$ onto W.

(b) Find the orthogonal decomposition of $\mathbf{v}$ with respect to W; in other words, find vectors $\mathbf{a}$ in W and $\mathbf{b}$ in $W^\perp$ such that $\mathbf{v} = \mathbf{a} + \mathbf{b}$.

(c) Find an orthonormal basis for W.

a) $\text{proj}_W(\vec{v}) = \text{proj}_{\vec{u}_1}(\vec{v}) + \text{proj}_{\vec{u}_2}(\vec{v}) + \text{proj}_{\vec{u}_3}(\vec{v})$

$$= \left(\frac{\vec{v}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\right)\vec{u}_1 + \left(\frac{\vec{v}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\right)\vec{u}_2 + \left(\frac{\vec{v}\cdot\vec{u}_3}{\vec{u}_3\cdot\vec{u}_3}\right)\vec{u}_3$$

$$= \frac{14}{7}\vec{u}_1 + \frac{21}{21}\vec{u}_2 + \frac{-24}{6}\vec{u}_3 = 2\vec{u}_1 + \vec{u}_2 - 4\vec{u}_3$$

$$= 2\begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 3 \\ 2 \end{bmatrix} - 4\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ -8 \\ 7 \\ 4 \end{bmatrix}$$

b) $\vec{a} = \text{proj}_W(\vec{v}) = \begin{bmatrix} -4 \\ -8 \\ 7 \\ 4 \end{bmatrix}$

$$\vec{b} = \text{perp}_W(\vec{v}) = \vec{v} - \vec{a} = \begin{bmatrix} -7 \\ -8 \\ 7 \\ 1 \end{bmatrix} - \begin{bmatrix} -4 \\ -8 \\ 7 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 0 \\ -3 \end{bmatrix} \quad \therefore \vec{v} = \underbrace{\begin{bmatrix} -4 \\ -8 \\ 7 \\ 4 \end{bmatrix}}_{\text{in } W} + \underbrace{\begin{bmatrix} -3 \\ 0 \\ 0 \\ -3 \end{bmatrix}}_{\text{in } W^\perp}$$

c) $\|\vec{u}_1\| = \sqrt{7}$, $\|\vec{u}_2\| = \sqrt{21}$, $\|\vec{u}_3\| = \sqrt{6}$

$\therefore$ An orthonormal basis for W is

$$\left\{ \frac{1}{\sqrt{7}}\begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{21}}\begin{bmatrix} -2 \\ 2 \\ 3 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} \right\}$$