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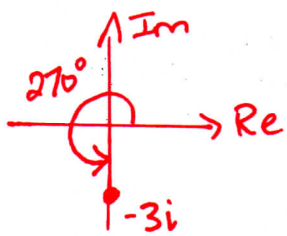
Name: SOLUTIONS

Mark: $\overline{}$ **25**

MATH 251
Assignment 5

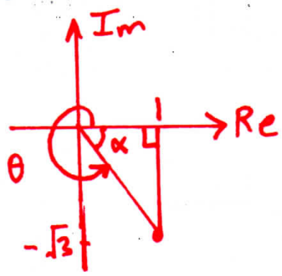
1. (1 mark) Write $-3i$ in phasor form using degrees and then evaluate the product $(-3i)(5\angle 25^\circ)$. Express your answer in phasor form.



$$-3i = 3\angle 270^\circ$$

$$\therefore\ (-3i)(5\angle 25^\circ) = (3\angle 270^\circ)(5\angle 25^\circ) = 15\angle 295^\circ$$

2. (3 marks) Evaluate $(1 - i\sqrt{3})^{14}$. Express your answer in both Euler and rectangular form, using exact values in both cases.



$$r = \sqrt{1^2 + (-\sqrt{3})^2} = 2$$

$$\alpha = \tan^{-1}\frac{\sqrt{3}}{1} = \frac{\pi}{3} \qquad \therefore\ \theta = 2\pi - \alpha = \frac{5\pi}{3} \quad \left(\text{or } -\frac{\pi}{3}\right)$$

$$\therefore\ 1 - i\sqrt{3} = 2e^{i\frac{5\pi}{3}}$$

$$(1 - i\sqrt{3})^{14} = \left(2e^{i\frac{5\pi}{3}}\right)^{14} = 2^{14}e^{i\frac{70\pi}{3}} = 16384\,e^{i\frac{70\pi}{3}} \text{ or } 16384\,e^{i\frac{4\pi}{3}}$$

(where $\frac{4\pi}{3}$ and $\frac{70\pi}{3}$ are coterminal angles; could also use $-\frac{2\pi}{3}$)

In rectangular form

$$16384\,e^{i\frac{4\pi}{3}} = 16384\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = 16384\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$= -8192 - 8192i\sqrt{3}$$

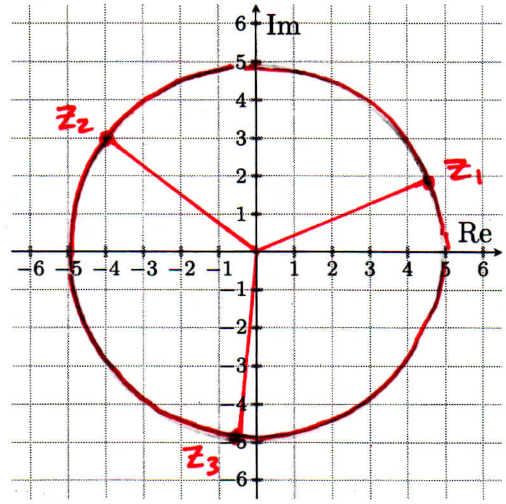
3. (4 marks) Find all the cube roots of $44 + 117i$. Express each root in the rectangular form $a + bi$ with a and b rounded to three decimal places. Plot each of the roots in the complex plane.

Let $z^3 = 44 + 117i$

$r = \sqrt{44^2 + 117^2} = 125$

$\theta = \tan^{-1}\left(\frac{117}{44}\right) \approx 1.21109$

$\therefore\ z^3 = 125e^{1.21109i}$



1st root is $z_1 = \sqrt[3]{125}\, e^{\frac{1.21109}{3} i} = 5e^{0.4037i}$

$= 5(\cos 0.4037 + i \sin 0.4037) = 4.598 + 1.964i$

2nd root is $z_2 = \sqrt[3]{125}\, e^{(0.4037 + \frac{2\pi}{3})i} = 5e^{2.4981i}$

$= 5(\cos 2.4981 + i \sin 2.4981) = -4 + 3i$

3rd root is $z_3 = \sqrt[3]{125}\, e^{(0.4037 + \frac{4\pi}{3})i} = 5e^{4.5925i}$

$= 5(\cos 4.5925 + i \sin 4.5925) = -0.598 - 4.964i$

4. (2 marks) Suppose $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$. Express the quotient $z = z_1/z_2$ in the form $a + bi$.

$$z = \frac{a_1 + b_1 i}{a_2 + b_2 i} \cdot \frac{a_2 - b_2 i}{a_2 - b_2 i} = \frac{a_1 a_2 - a_1 b_2 i + a_2 b_1 i + b_1 b_2}{a_2^2 + b_2^2} \quad (\text{using } i^2 = -1)$$

$$= \left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}\right) + \left(\frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}\right) i$$

5. (2 marks) Show that $\mathbf{v} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ and find the corresponding eigenvalue.

$$A\vec{v} = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ $\vec{v}$ is an eigenvector of A with associated eigenvalue $\lambda = 2$

6. (4 marks) Suppose $\lambda_1 = 3$ is an eigenvalue of $A = \begin{bmatrix} 4 & 5 \\ -2 & k \end{bmatrix}$.

(a) Find k.

(b) Find the other eigenvalue, λ_2.

(c) Find one eigenvector associated with each eigenvalue λ_1 and λ_2.

a) $\det(A-\lambda_1 I) = \begin{vmatrix} 4-3 & 5 \\ -2 & K-3 \end{vmatrix} = \begin{vmatrix} 1 & 5 \\ -2 & K-3 \end{vmatrix} = (K-3)+10 = K+7 = 0 \Rightarrow K=-7$

b) $A = \begin{bmatrix} 4 & 5 \\ -2 & -7 \end{bmatrix}$ $\det(A-\lambda I) = \begin{vmatrix} 4-\lambda & 5 \\ -2 & -7-\lambda \end{vmatrix} = (4-\lambda)(-7-\lambda)+10$

$= \lambda^2+3\lambda-28+10 = \lambda^2+3\lambda-18 = (\lambda-3)(\lambda+6) = 0$

$\Rightarrow \lambda = 3$ or -6 $\therefore \lambda_2 = -6$

c) $\underline{\lambda_1=3}$ $A-\lambda_1 I = \begin{bmatrix} 1 & 5 \\ -2 & -10 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 5 & 0 \\ -2 & -10 & 0 \end{array}\right] \xrightarrow{R_2+2R_1} \left[\begin{array}{cc|c} 1 & 5 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x = -5t,\ y = t$

$\vec{x} = \begin{bmatrix} -5t \\ t \end{bmatrix} = t\begin{bmatrix} -5 \\ 1 \end{bmatrix}$ $\therefore \vec{x}_1 = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_1 = 3$

$\underline{\lambda_2=-6}$ $A-\lambda_2 I = \begin{bmatrix} 10 & 5 \\ -2 & -1 \end{bmatrix}$

$\left[\begin{array}{cc|c} 10 & 5 & 0 \\ -2 & -1 & 0 \end{array}\right] \xrightarrow[R_2+\frac{1}{5}R_1]{\frac{1}{10}R_1} \left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x = -\frac{1}{2}t,\ y = t$

$\vec{x} = \begin{bmatrix} -\frac{1}{2}t \\ t \end{bmatrix} = \frac{1}{2}t\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $\therefore \vec{x}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is an eigenvector for $\lambda_2 = -6$

7. (3 marks) Use Cramer's Rule to solve the following system.

$$\begin{cases} x_1 - 3x_2 + x_3 = 4 \\ 2x_1 - x_2 = -2 \\ 4x_1 - 3x_3 = 0 \end{cases}$$

$$A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -1 & 0 \\ 4 & 0 & -3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix}$$

$$A_1(\vec{b}) = \begin{bmatrix} 4 & -3 & 1 \\ -2 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad A_2(\vec{b}) = \begin{bmatrix} 1 & 4 & 1 \\ 2 & -2 & 0 \\ 4 & 0 & -3 \end{bmatrix}, \quad A_3(\vec{b}) = \begin{bmatrix} 1 & -3 & 4 \\ 2 & -1 & -2 \\ 4 & 0 & 0 \end{bmatrix}$$

$$\det(A) = 4\begin{vmatrix} -3 & 1 \\ -1 & 0 \end{vmatrix} + (-3)\begin{vmatrix} 1 & -3 \\ 2 & -1 \end{vmatrix} = 4(1) + (-3)(5) = -11$$

$$\det(A_1(\vec{b})) = (-3)\begin{vmatrix} 4 & -3 \\ -2 & -1 \end{vmatrix} = (-3)(-10) = 30$$

$$\det(A_2(\vec{b})) = 4\begin{vmatrix} 4 & 1 \\ -2 & 0 \end{vmatrix} + (-3)\begin{vmatrix} 1 & 4 \\ 2 & -2 \end{vmatrix} = 4(2) + (-3)(-10) = 38$$

$$\det(A_3(\vec{b})) = 4\begin{vmatrix} -3 & 4 \\ -1 & -2 \end{vmatrix} = 4(10) = 40$$

$$\therefore \ x_1 = \frac{30}{-11}, \quad x_2 = \frac{38}{-11}, \quad x_3 = \frac{40}{-11}$$

$$\vec{x} = \begin{bmatrix} -30/11 \\ -38/11 \\ -40/11 \end{bmatrix} \quad \text{or} \quad -\frac{2}{11}\begin{bmatrix} 15 \\ 19 \\ 20 \end{bmatrix}$$

8. (4 marks) Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & \tan\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}.$$

(a) Find and simplify $\det(A)$.

(b) Use the cofactor method to find A^{-1}.

a) $\det(A) = 1 \cdot \begin{vmatrix} 1 & \tan\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos\theta + \tan\theta \cdot \sin\theta$

$= \cos\theta + \dfrac{\sin^2\theta}{\cos\theta} = \dfrac{\cos^2\theta + \sin^2\theta}{\cos\theta} = \dfrac{1}{\cos\theta} = \sec\theta$

b)

$$C = \begin{bmatrix} +\begin{vmatrix} 1 & 0 \\ 0 & \cos\theta \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ -\sin\theta & \cos\theta \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ -\sin\theta & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & \tan\theta \\ 0 & \cos\theta \end{vmatrix} & +\begin{vmatrix} 1 & \tan\theta \\ -\sin\theta & \cos\theta \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ -\sin\theta & 0 \end{vmatrix} \\ +\begin{vmatrix} 0 & \tan\theta \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & \tan\theta \\ 0 & 0 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & \cos\theta + \sin\theta\tan\theta & 0 \\ -\tan\theta & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & \sec\theta & 0 \\ -\tan\theta & 0 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{\det(A)} C^T = \cos\theta \begin{bmatrix} \cos\theta & 0 & -\tan\theta \\ 0 & \sec\theta & 0 \\ \sin\theta & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos^2\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta\cos\theta & 0 & \cos\theta \end{bmatrix}$$

9. (2 marks) Use determinants to find the volume of the parallelepiped formed by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ 1 \\ -7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -6 \\ 0 \end{bmatrix}.$$

$$\begin{vmatrix} 1 & -2 & 3 \\ 5 & 1 & -7 \\ 1 & -6 & 0 \end{vmatrix} = 1 \begin{vmatrix} -2 & 3 \\ 1 & -7 \end{vmatrix} - (-6) \begin{vmatrix} 1 & 3 \\ 5 & -7 \end{vmatrix} = 1(11) + 6(-22) = -121$$

$$\therefore \text{Volume} = |-121| = 121$$