CAMOSUN

Name: SOLUTIONS

Mark: $\overline{}$ **25**

MATH 251
Assignment 4

1. (3 marks) Find an LU factorization of $A = \begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ -6 & -4 & 0 \end{bmatrix}$.

$$A \to \begin{matrix} \\ R_2+R_1 \\ R_3+2R_1 \end{matrix} \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & -18 & -4 \end{bmatrix} \to \begin{matrix} \\ \\ R_3-9R_2 \end{matrix} \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & 5 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 9 & 1 \end{bmatrix} \qquad \therefore A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 9 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & 5 \end{bmatrix}$$

2. (3 marks) Solve the following system using the LU method.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 3 & 7 & 2 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}}_{\vec{b}}$$

Let $\vec{y} = U\vec{x}$

(i) Solve $L\vec{y} = \vec{b}$ $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$

$y_1 = 4$

$2y_1 + y_2 = -2 \Rightarrow 8 + y_2 = -2 \Rightarrow y_2 = -10$

$-y_1 + y_2 + y_3 = 1 \Rightarrow -4 + (-10) + y_3 = 1 \Rightarrow y_3 = 15$

(ii) Solve $U\vec{x} = \vec{y}$ $\begin{bmatrix} 3 & 7 & 2 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ 15 \end{bmatrix}$

$5x_3 = 15 \Rightarrow x_3 = 3$

$5x_2 = -10 \Rightarrow x_2 = -2$

$3x_1 + 7x_2 + 2x_3 = 4 \Rightarrow 3x_1 + (-14) + 6 = 4 \Rightarrow 3x_1 = 12 \Rightarrow x_1 = 4$

$\therefore \vec{x} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$

3. (2 marks) Let S be the set of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ satisfying the property $y = \pm x$. Either prove that S forms a subspace of $\mathbb{R}^2$ or give a counterexample to show that it does not.

$\vec{0}$ is in S and S is closed under scalar multiplication. However, S is not a subspace of $\mathbb{R}^2$ since it's not closed under addition. Consider $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Both $\vec{u}$ and $\vec{v}$ are in S, but $\vec{u} + \vec{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ is not in S since $0 \neq \pm 2$.

4. (3 marks) Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -2 & -5 & -7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -4 \\ 3 \\ -1 \end{bmatrix}$. Determine whether or not $\mathbf{v}$ belongs to the following subspaces of $\mathbb{R}^3$.

(a) col(A)

(b) null(A)

a) Solve $c_1 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 7 \\ -5 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 9 \\ -7 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ -1 \end{bmatrix}$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & -4 \\ 3 & 7 & 9 & 3 \\ -2 & -5 & -7 & -1 \end{array}\right] \xrightarrow{\substack{R_2 - 3R_1 \\ R_3 + 2R_1}} \left[\begin{array}{ccc|c} 1 & 2 & 2 & -4 \\ 0 & 1 & 3 & 15 \\ 0 & -1 & -3 & -9 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 2 & 2 & -4 \\ 0 & 1 & 3 & 15 \\ 0 & 0 & 0 & 6 \end{array}\right]$$

← system is inconsistent (no solution)

$\therefore$ $\vec{v}$ is <u>not</u> in col(A).

b) $A\vec{v} = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -2 & -5 & -7 \end{bmatrix} \begin{bmatrix} -4 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$

$\therefore$ $\vec{v}$ is in null(A).

5. (5 marks) Suppose

$$A = \begin{bmatrix} 1 & -3 & 2 & 1 \\ -2 & 6 & -5 & 1 \\ 4 & -12 & 7 & 7 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -3 & 0 & 7 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & -2 & 4 \\ -3 & 6 & -12 \\ 2 & -5 & 7 \\ 1 & 1 & 7 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for the column space of A consisting of columns of A.

(b) Express each column vector of A that is not in your basis from part (a) as a linear combination of basis vectors.

(c) Find a basis for the row space of A consisting of rows of A.

(d) Find a basis for the null space of A.

(e) Find rank(A) and nullity(A).

a) $\left\{ \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 7 \end{bmatrix} \right\}$

b) $\begin{bmatrix} -3 \\ 6 \\ -12 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ -5 \\ 7 \end{bmatrix}$

c) $\{ [1 \ -3 \ 2 \ 1], [-2 \ 6 \ -5 \ 1] \}$

d) Solving $A\vec{x} = \vec{0}$ using RREF of A gives

$x_2 = s$, $x_4 = t$, $x_1 = 3s - 7t$, $x_3 = 3t$

i.e. $\vec{x} = \begin{bmatrix} 3s-7t \\ s \\ 3t \\ t \end{bmatrix} = s \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -7 \\ 0 \\ 3 \\ 1 \end{bmatrix}$ $\therefore$ basis is $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$

e) rank(A) = 2 and nullity(A) = 2

6. (3 marks) Let $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ satisfying $T(\mathbf{i}) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $T(\mathbf{j}) = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

   (a) Find $T(\mathbf{v})$.

   (b) Find $T^{-1}(\mathbf{v})$.

a) Standard matrix for T is $A = \begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}$.

$\therefore T(\vec{v}) = A\vec{v} = \begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ -6 \end{bmatrix}$.

b) $A^{-1} = \frac{1}{-1}\begin{bmatrix} 5 & -4 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ 4 & -3 \end{bmatrix}$.

$\therefore T^{-1}(\vec{v}) = A^{-1}\vec{v} = \begin{bmatrix} -5 & 4 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -13 \\ 10 \end{bmatrix}$.

7. (3 marks) Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

(a) Find the coordinate vector $[\mathbf{w}]_{\mathcal{B}}$ of $\mathbf{w}$ with respect to the basis $\mathcal{B} = \{\mathbf{u}, \mathbf{v}\}$ of $\mathbb{R}^2$.

(b) If $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation and

$$T(\mathbf{u}) = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \quad \text{and} \quad T(\mathbf{v}) = \begin{bmatrix} 4 \\ 8 \\ -1 \end{bmatrix},$$

then find $T(\mathbf{w})$.

a) $\left[\begin{array}{cc|c} 1 & 3 & 4 \\ 2 & 4 & 2 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & -2 & -6 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 1 & 3 \end{array}\right]$

$\xrightarrow{R_1 - 3R_2} \left[\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & 3 \end{array}\right]$ $\therefore \vec{w} = -5\vec{u} + 3\vec{v}$ and $[\vec{w}]_B = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$.

b) $T(\vec{w}) = T(-5\vec{u} + 3\vec{v}) = -5T(\vec{u}) + 3T(\vec{v})$

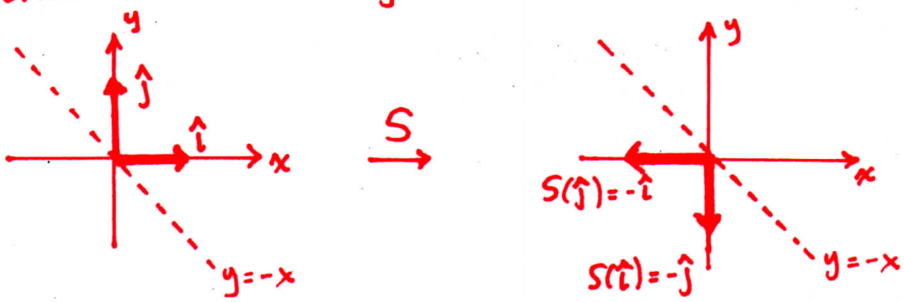
$= -5\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 4 \\ 8 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 34 \\ -18 \end{bmatrix}$.

8. (3 marks) Consider the linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ that reflects a vector about the line $y = -x$, then rotates it clockwise by $30°$, and then finally projects it onto the y-axis.

(a) Find the standard matrix of T.

(b) Use your answer from part (a) to find $T(\mathbf{v})$, where $\mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$. Give an exact answer.

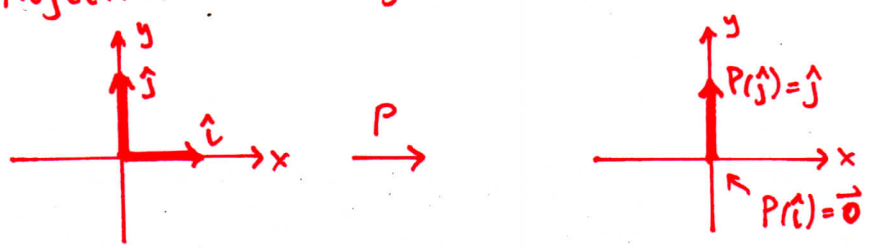
a) Reflection S about $y=-x$.



$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

Rotation $R_{-30°}$ $B = \begin{bmatrix} \cos(-30°) & -\sin(-30°) \\ \sin(-30°) & \cos(-30°) \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$

Projection P onto y-axis.



$C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Matrix for $T = P \circ R_{-30°} \circ S$ is

$$D = CBA = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

b) $$T(\vec{v}) = D\vec{v} = \begin{bmatrix} 0 & 0 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{3}+2 \end{bmatrix}.$$