CAMOSUN

Name: SOLUTIONS

Mark: $\frac{\quad}{25}$

MATH 251
Assignment 3

1. (2 marks) Suppose A is a 2×4 matrix and B is a 3×2 matrix. Find the size of matrix C in each equation, assuming the matrix operations are well-defined.
   (a) $ACB = I_2$
   (b) $4A^T - 3C = O$

a) $\underset{2\times 4}{A}\ \underset{4\times 3}{C}\ \underset{3\times 2}{B} = \underset{2\times 2}{I_2}$ $\quad$ C must be 4×3

b) $4A^T = 3C$ $\quad$ C must be 4×2 (same as A^T)

$C = \frac{4}{3}A^T$

2. (2 marks) Let $A = \begin{bmatrix} 1 & a \\ 2 & 1 \end{bmatrix}$. If $A^2 = 2A$, then find a.

$$A^2 = \begin{bmatrix} 1 & a \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & a \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+2a & 2a \\ 4 & 2a+1 \end{bmatrix}$$

$$2A = 2\begin{bmatrix} 1 & a \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2a \\ 4 & 2 \end{bmatrix}$$

$A^2 = 2A$ iff $2a+1 = 2$; i.e. $a = \frac{1}{2}$

3. (3 marks) Evaluate (if possible) $B^T B + 4A^2 - 3I_2$, where $A = \begin{bmatrix} 7 & 1 \\ -5 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 0 \\ 1 & 4 \\ 9 & -1 \end{bmatrix}$.

$$B^T B = \begin{bmatrix} 6 & 1 & 9 \\ 0 & 4 & -1 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 1 & 4 \\ 9 & -1 \end{bmatrix} = \begin{bmatrix} 118 & -5 \\ -5 & 17 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 7 & 1 \\ -5 & -4 \end{bmatrix} \begin{bmatrix} 7 & 1 \\ -5 & -4 \end{bmatrix} = \begin{bmatrix} 44 & 3 \\ -15 & 11 \end{bmatrix}$$

$$B^T B + 4A^2 - 3I_2 = \begin{bmatrix} 118 & -5 \\ -5 & 17 \end{bmatrix} + 4\begin{bmatrix} 44 & 3 \\ -15 & 11 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 118 & -5 \\ -5 & 17 \end{bmatrix} + \begin{bmatrix} 176 & 12 \\ -60 & 44 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 291 & 7 \\ -65 & 58 \end{bmatrix}$$

4. (3 marks) Write D as a linear combination of A, B, and C (if possible), where

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} -4 & -8 \\ -3 & 9 \end{bmatrix}.$$

Solve $c_1 A + c_2 B + c_3 C = D$. Writing entries of 2×2 matrices as 4×1 vectors and solving gives:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -4 \\ -1 & 1 & 2 & -8 \\ -1 & 1 & 1 & -3 \\ 3 & 2 & 1 & 9 \end{array}\right] \xrightarrow[\substack{R_3+R_1 \\ R_4-3R_1}]{R_2+R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -4 \\ 0 & 2 & 4 & -12 \\ 0 & 2 & 3 & -7 \\ 0 & -1 & -5 & 21 \end{array}\right]$$

$$\xrightarrow[\substack{\frac{1}{2}R_2 \\ R_3-R_2 \\ R_4+\frac{1}{2}R_2}]{R_1-\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & -6 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & -3 & 15 \end{array}\right] \xrightarrow[\substack{-R_3 \\ R_4-3R_3}]{R_2+2R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ $c_1 = 2$, $c_2 = 4$, $c_3 = -5$

$\therefore$ $D = 2A + 4B - 5C$

5. (3 marks) Determine whether AB is in span(A, B), where $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 4 \\ 0 & 3 \end{bmatrix}$.

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 10 \\ 0 & 9 \end{bmatrix}$$

Solve $c_1 A + c_2 B = AB$. Writing entries of 2×2 matrices as 4×1 vectors gives:

$$\left[\begin{array}{cc|c} 1 & -1 & -1 \\ 2 & 4 & 10 \\ 0 & 0 & 0 \\ 3 & 3 & 9 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_4} \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 2 & 4 & 10 \\ 3 & 3 & 9 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 6 & 12 \\ 0 & 6 & 12 \\ 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\substack{R_1 + \frac{1}{6}R_2 \\ \frac{1}{6}R_2 \\ R_3 - R_2}} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore$ $c_1 = 1$ and $c_2 = 2$ and

$$AB = A + 2B$$

Since AB is a linear combination of A and B, then it is in span(A, B).

6. (3 marks) If $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 2 & 5 \end{bmatrix}$, then solve the following equation for X.

$$XAA^T + B - 3I_2 = O$$

$$X(AA^T) = 3I_2 - B \Rightarrow X = (3I_2 - B)(AA^T)^{-1}$$

$$3I_2 - B = 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -2 & -2 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 2 & -2 & 3 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -2 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 17 & 10 \\ 10 & 6 \end{bmatrix}$$

$$(AA^T)^{-1} = \frac{1}{2}\begin{bmatrix} 6 & -10 \\ -10 & 17 \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} 2 & 0 \\ -2 & -2 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 6 & -10 \\ -10 & 17 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 6 & -10 \\ -10 & 17 \end{bmatrix} = \begin{bmatrix} 6 & -10 \\ 4 & -7 \end{bmatrix}$$

7. (2 marks) Let $A = \begin{bmatrix} 1 & 4 & 0 \\ -2 & 2 & -3 \\ 5 & 1 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 1 & 7 \\ -2 & 2 & -3 \\ 1 & 4 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 10 & -3 \\ 5 & 1 & 7 \end{bmatrix}$.

(a) Find an elementary matrix E satisfying the equation $EA = B$.

(b) Find an elementary matrix E satisfying the equation $EA = C$.

a) $E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ Corresponding to $R_1 \leftrightarrow R_3$

b) $E = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Corresponding to $R_2 + 2R_1$

8. (1 mark) Find the inverse of $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

$$A^{-1} = \sqrt{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \sqrt{2} \cdot \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\text{or } \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

9. (3 marks) Let $A = \begin{bmatrix} 1 & 0 \\ -4 & 3 \end{bmatrix}$. Write A^{-1} and A each as a product of two elementary matrices.

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ -4 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R_2 + 4R_1} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 3 & 4 & 1 \end{array}\right] \quad E_1 = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 4/3 & 1/3 \end{array}\right] \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 4/3 & 1/3 \end{bmatrix}}_{A^{-1}}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ 4/3 & 1/3 \end{bmatrix} = E_2 E_1 I = \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$$

$$\text{and } A = E_1^{-1} E_2^{-1} = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

10. (3 marks) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 7 \\ 5 \end{bmatrix}$.

(a) Find A^{-1} using the Gauss-Jordan method for computing the inverse.

(b) Use A^{-1} to solve the system $A\mathbf{x} = \mathbf{b}$.

a) $$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \to \begin{array}{c} \\ R_2-2R_1 \\ R_3-R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

$$\to \begin{array}{c} R_1-2R_2 \\ \\ R_3+2R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \to \begin{array}{c} R_1+9R_3 \\ R_2-3R_3 \\ -R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

b) $$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \\ 5 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$