CAMOSUN

Name: SOLUTIONS

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MATH 251
Assignment 2

1. (3 marks) Consider the following system of linear equations.

$$\begin{cases} 6x_1 - 3x_2 + 2x_4 = 7 \\ x_2 + 2x_3 + 5x_4 = 15 \\ 3x_3 - x_4 = -5 \\ 3x_4 = 6 \end{cases}$$

   (a) Construct the augmented matrix for this system. Is it in row echelon form (REF)? Is it in reduced row echelon form (RREF)?

   (b) Solve the system by using back substitution.

a) $\left[\begin{array}{cccc|c} 6 & -3 & 0 & 2 & 7 \\ 0 & 1 & 2 & 5 & 15 \\ 0 & 0 & 3 & -1 & -5 \\ 0 & 0 & 0 & 3 & 6 \end{array}\right]$ REF but not RREF

b) $x_4 = 2$

$3x_3 - 2 = -5 \Rightarrow 3x_3 = -3 \Rightarrow x_3 = -1$

$x_2 + 2(-1) + 5(2) = 15 \Rightarrow x_2 = 7$

$6x_1 - 3(7) + 2(2) = 7 \Rightarrow 6x_1 = 24 \Rightarrow x_1 = 4$

$\therefore \vec{x} = \begin{bmatrix} 4 \\ 7 \\ -1 \\ 2 \end{bmatrix}$

2. (3 marks) Solve the system of equations using the Gauss-Jordan method.

$$\begin{cases} x - 2y + z = 0 \\ 2x - 2y - 6z = 8 \\ -4x + 5y + 9z = -9 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & -2 & -6 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \xrightarrow[R_3+4R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\xrightarrow[\substack{\frac{1}{2}R_2 \\ R_3+\frac{3}{2}R_2}]{R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[R_2+4R_3]{R_1+7R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \text{RREF}$$

$\therefore\ x = 29,\ y = 16,\ z = 3$

3. (2 marks) For what value(s) of k (if any) is the system of linear equations having the following augmented matrix inconsistent?

$$\left[\begin{array}{cc|c} 1 & k & 2 \\ k & 4 & 4 \end{array}\right]$$

$$\xrightarrow{R_2-kR_1} \left[\begin{array}{cc|c} 1 & k & 2 \\ 0 & 4-k^2 & 4-2k \end{array}\right] \quad \text{REF}$$

Need $4-k^2=0$ (i.e. $k=\pm 2$) and $4-2k \neq 0$ (i.e. $k \neq 2$).

$\therefore\ k=-2$

4. (5 marks) Consider the two planes $x + 3y - 2z = 2$ and $3x - 5y + 8z = -8$.
   (a) Find parametric equations for the line of intersection of these two planes.
   (b) Determine whether the line found in part (a) and the line described by

$$x = 6 + 3t, \quad y = -9 - 4t, \quad z = 5 + t$$

intersect, and if they do, find their point of intersection.

a) $\left[\begin{array}{ccc|c} 1 & 3 & -2 & 2 \\ 3 & -5 & 8 & -8 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 2 \\ 0 & -14 & 14 & -14 \end{array}\right]$

$\xrightarrow{-\frac{1}{14}R_2} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 2 \\ 0 & 1 & -1 & 1 \end{array}\right] \xrightarrow{R_1 - 3R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right]$

$z = t, \; x = -1 - t, \; y = 1 + t \quad \therefore \begin{cases} x = -1 - t \\ y = 1 + t \\ z = t \end{cases}$

b) Solve
$$\begin{aligned} -1 - t_1 &= 6 + 3t_2 \\ 1 + t_1 &= -9 - 4t_2 \\ t_1 &= 5 + t_2 \end{aligned} \quad \rightarrow \quad \begin{aligned} -t_1 - 3t_2 &= 7 \\ t_1 + 4t_2 &= -10 \\ t_1 - t_2 &= 5 \end{aligned}$$

$\left[\begin{array}{cc|c} -1 & -3 & 7 \\ 1 & 4 & -10 \\ 1 & -1 & 5 \end{array}\right] \xrightarrow{\substack{-R_1 \\ R_2 + R_1 \\ R_3 + R_1}} \left[\begin{array}{cc|c} 1 & 3 & -7 \\ 0 & 1 & -3 \\ 0 & -4 & 12 \end{array}\right] \xrightarrow{\substack{R_1 - 3R_2 \\ R_3 + 4R_2}} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right]$

$\therefore \; t_1 = 2$ and $t_2 = -3$

$x = -1 - 2 = -3$ $\qquad$ $x = 6 + 3(-3) = -3$
$y = 1 + 2 = 3$ $\qquad$ $y = -9 - 4(-3) = 3$
$z = 2$ $\qquad$ $z = 5 + (-3) = 2$

$\therefore$ Lines intersect at the point $(-3, 3, 2)$.

5. (3 marks) Consider the vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}.$$

(a) Determine whether **u**, **v**, and **w** are linearly independent or linearly dependent. If they are linearly dependent, find a dependent relationship among the vectors.

(b) Determine whether or not **v** is in span(**u**, **w**).

a) Solve $c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 0 \\ 3 & 4 & 5 & 0 \\ 4 & 5 & 6 & 0 \end{array}\right] \xrightarrow{\substack{R_2-2R_1 \\ R_3-3R_1 \\ R_4-4R_1}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & -2 & -4 & 0 \\ 0 & -3 & -6 & 0 \end{array}\right] \xrightarrow{\substack{R_1+2R_2 \\ -R_2 \\ R_3-2R_2 \\ R_4-3R_2}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$c_3 = t, \; c_2 = -2t, \; c_1 = t$

Since there are nontrivial solutions (eg $c_1 = 1, c_2 = -2, c_3 = 1$ when $t = 1$) then vectors are L. D. In particular

$$\vec{u} - 2\vec{v} + \vec{w} = \vec{0}$$

b) $\vec{u} - 2\vec{v} + \vec{w} = \vec{0} \Rightarrow \vec{v} = \frac{1}{2}\vec{u} + \frac{1}{2}\vec{w}$

Yes, $\vec{v}$ is in span$(\vec{u}, \vec{w})$ since it is a linear combination of $\vec{u}$ and $\vec{w}$.

6. (5 marks) A supermarket sells three different bags of assorted fruit, each containing a mixture of bananas, apples and oranges: *Awesome Fruits*™, *Bodacious Fruits*™ and *Comfort Fruits*™. The number of each fruit in each bag is summarized in the following table.

	Awesome Fruits (AF)	Bodacious Fruits (BF)	Comfort Fruits (CF)
bananas	2	6	2
apples	3	10	4
oranges	2	5	1

Can a customer buy bags of fruit from the supermarket so as to get exactly 16 bananas, 25 apples and 15 oranges? If so, how many bags of each of the three varieties must he or she buy? If there is more than one possibility, then find all the solutions. It it's not possible, then explain why. To support your answer, set up and solve a system of linear equations.

Let x_1, x_2, x_3 be the number of bags of AF, BF, CF respectively.

We want
$$\begin{aligned} 2x_1 + 6x_2 + 2x_3 &= 16 \quad \text{(bananas)} \\ 3x_1 + 10x_2 + 4x_3 &= 25 \quad \text{(apples)} \\ 2x_1 + 5x_2 + x_3 &= 15 \quad \text{(oranges)} \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & 6 & 2 & 16 \\ 3 & 10 & 4 & 25 \\ 2 & 5 & 1 & 15 \end{array}\right] \xrightarrow[\substack{R_2 - \frac{3}{2}R_1 \\ R_3 - R_1}]{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 8 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{array}\right] \xrightarrow[R_3 + R_2]{R_1 - 3R_2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore x_1 = 5 + 2t,\ x_2 = 1 - t$ and $x_3 = t$

But we need $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$ and integer values

$x_1 = 5 + 2t \geq 0 \Rightarrow t \geq -\frac{5}{2}$, $x_2 = 1 - t \geq 0 \Rightarrow t \leq 1$, $x_3 = t \geq 0$

$\therefore$ $0 \leq t \leq 1$ and since t must be an integer (so that x_1, x_2, x_3 values are), then $t = 0$ or $t = 1$

Two solutions! If $t = 0$, $\vec{x} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$ and if $t = 1$, $\vec{x} = \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix}$.

$\therefore$ The customer must buy 5 bags of AF and 1 bag of BF

<u>or</u> 7 bags of AF and 1 bag of CF.

7. (4 marks) Consider the unbalanced chemical equation for the following redox reaction involving the nine elements Cu, S, C, N, K, I, O, H, and Cl:

$$CuSCN + KIO_3 + HCl \longrightarrow CuSO_4 + KCl + HCN + ICl + H_2O$$

To balance the equation one must find positive integers $x_1, x_2, \ldots, x_8$ so that

$$x_1 CuSCN + x_2 KIO_3 + x_3 HCl \longrightarrow x_4 CuSO_4 + x_5 KCl + x_6 HCN + x_7 ICl + x_8 H_2O$$

has the same number of each atom on both sides.

(a) Set up a homogeneous system of linear equations for the variables $x_1, x_2, \ldots, x_8$.

(b) Construct the augmented matrix associated with the system of equations in part (a).

a)
Cu: $x_1 = x_4$
S: $x_1 = x_4$
C: $x_1 = x_6$
N: $x_1 = x_6$
K: $x_2 = x_5$
I: $x_2 = x_7$
O: $3x_2 = 4x_4 + x_8$
H: $x_3 = x_6 + 2x_8$
Cl: $x_3 = x_5 + x_7$

$\Rightarrow$

$x_1 - x_4 = 0$
$x_1 - x_4 = 0$
$x_1 - x_6 = 0$
$x_1 - x_6 = 0$
$x_2 - x_5 = 0$
$x_2 - x_7 = 0$
$3x_2 - 4x_4 - x_8 = 0$
$x_3 - x_6 - 2x_8 = 0$
$x_3 - x_5 - x_7 = 0$

b)

$$\left[\begin{array}{cccccccc|c}
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 3 & 0 & -4 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & -2 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0
\end{array}\right]$$

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(c) Find the RREF of the augmented matrix in part (b) by using the **rref** command in Octave or MATLAB.

(d) Use the RREF from part (c) to solve the system of equations and then balance the chemical equation.

c) RREF is

$$\left[\begin{array}{cccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -4/5 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -7/5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -14/5 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -4/5 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -7/5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -4/5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -7/5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

d) Let $x_8 = t$ (free variable).

Then $x_1 = \frac{4}{5}t$, $x_2 = \frac{7}{5}t$, $x_3 = \frac{14}{5}t$, $x_4 = \frac{4}{5}t$,
$x_5 = \frac{7}{5}t$, $x_6 = \frac{4}{5}t$, $x_7 = \frac{7}{5}t$, $x_8 = t$

Let $t = 5$ for smallest positive integer values.

Then $x_1 = 4$, $x_2 = 7$, $x_3 = 14$, $x_4 = 4$
$x_5 = 7$, $x_6 = 4$, $x_7 = 7$, $x_8 = 5$

$\therefore\ 4\,CuSCN + 7KIO_3 + 14\,HCl \rightarrow 4CuSO_4 + 7KCl + 4HCN + 7ICl + 5H_2O$