CAMOSUN

Name: SOLUTIONS

Mark: $\overline{}$ / 25

MATH 251
Assignment 1

1. (4 marks) Consider two points A and B in $\mathbb{R}^3$. Suppose $A = (3, 0, 2)$ and $\overrightarrow{AB} = \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$.

   (a) Find the coordinates of point B.

   (b) Find two distinct unit vectors that are parallel to $\overrightarrow{AB}$.

   (c) Find parametric equations for the line passing through points A and B.

   (d) At what point does the line from part (c) intersect the xy-plane?

a) $\overrightarrow{AB} = \vec{B} - \vec{A} \Rightarrow \vec{B} = \overrightarrow{AB} + \vec{A} = \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 8 \end{bmatrix}$ $\therefore B = (5, -3, 8)$

b) $\|\overrightarrow{AB}\| = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{4+9+36} = \sqrt{49} = 7$

$$\therefore \vec{u} = \pm \frac{1}{\|\overrightarrow{AB}\|} \overrightarrow{AB} = \pm \frac{1}{7} \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$$

c) $\vec{x} = \vec{A} + t\,\overrightarrow{AB} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix} \Rightarrow \begin{cases} x = 3 + 2t \\ y = -3t \\ z = 2 + 6t \end{cases}$ (Answer not unique.)

d) $z = 0 \Rightarrow 2 + 6t = 0 \Rightarrow t = -\frac{1}{3}$ $\therefore x = 3 + 2(-\frac{1}{3}) = \frac{7}{3}$ and $y = -3(-\frac{1}{3}) = 1$

$\therefore$ point is $(\frac{7}{3}, 1, 0)$

2. (7 marks) Consider the three points

$$A = (2, 0, -5), \qquad B = (8, 2, -9), \qquad C = (7, 9, -7).$$

(a) Find the area of triangle $\triangle ABC$. Give an exact, simplified answer.

(b) Find the angle $0° \leq \theta \leq 180°$ between $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Round your answer to two decimal places.

(c) Determine whether or not $\triangle ABC$ is a right triangle.

(d) Find the equation, in general form, of the plane passing through the points A, B, and C.

a) $\overrightarrow{AB} = \begin{bmatrix} 8 \\ 2 \\ -9 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ -4 \end{bmatrix}$ and $\overrightarrow{AC} = \begin{bmatrix} 7 \\ 9 \\ -7 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ -2 \end{bmatrix}$

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6 & 2 & -4 \\ 5 & 9 & -2 \end{vmatrix} \begin{matrix} \hat{i} & \hat{j} \\ 6 & 2 \\ 5 & 9 \end{matrix} = -4\hat{i} - 20\hat{j} + 54\hat{k} - 10\hat{k} + 36\hat{i} + 12\hat{j}$

$= 32\hat{i} - 8\hat{j} + 44\hat{k}$ $\quad \therefore \text{Area} = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{1}{2}\sqrt{32^2 + (-8)^2 + 44^2}$

$= \frac{1}{2}\sqrt{3024} = \frac{1}{2}(12\sqrt{21}) = 6\sqrt{21}$

b) $\|\overrightarrow{AB}\| = \sqrt{6^2 + 2^2 + (-4)^2} = \sqrt{56} = 2\sqrt{14}$

$\|\overrightarrow{AC}\| = \sqrt{5^2 + 9^2 + (-2)^2} = \sqrt{110}$

$\overrightarrow{AB} \cdot \overrightarrow{AC} = 30 + 18 + 8 = 56$

$\cos\theta = \dfrac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\|\overrightarrow{AB}\| \|\overrightarrow{AC}\|} = \dfrac{56}{2\sqrt{14}\sqrt{110}} = \dfrac{14}{\sqrt{385}} \Rightarrow \theta = \cos^{-1}\dfrac{14}{\sqrt{385}} \approx 44.48°$

c) $\overrightarrow{BC} = \begin{bmatrix} 7 \\ 9 \\ -7 \end{bmatrix} - \begin{bmatrix} 8 \\ 2 \\ -9 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix}$ $\quad \overrightarrow{AB} \cdot \overrightarrow{BC} = -6 + 14 - 8 = 0$ $\quad \therefore \overrightarrow{AB} \perp \overrightarrow{BC}$

and $\angle B = 90°$ and so $\triangle ABC$ <u>is</u> a right triangle.

d) $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{bmatrix} 32 \\ -8 \\ 44 \end{bmatrix}$ $\quad \vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{A}$

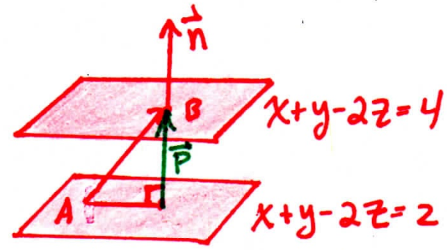
$32x - 8y + 44z = 32(2) - 8(0) + 44(-5)$

$32x - 8y + 44z = -156$

or $\quad 8x - 2y + 11z = -39$

3. (3 marks) Using projections, find the distance between the parallel planes $x + y - 2z = 2$ and $x + y - 2z = 4$. Given an exact, simplified answer.

Let A be a point on 1^{st} plane, e.g. $A = (0,0,-1)$.
Let B be a point on 2^{nd} plane, eg $B = (0,0,-2)$



Then $\vec{AB} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$. Also $\vec{n} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ is a normal vector to both planes.

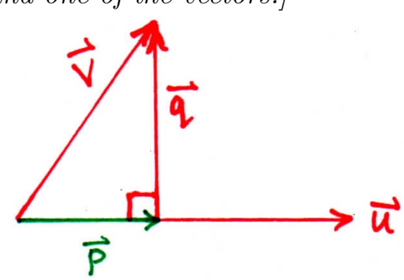
Let $\vec{p} = \text{proj}_{\vec{n}}(\vec{AB}) = \left(\frac{\vec{n} \cdot \vec{AB}}{\vec{n} \cdot \vec{n}}\right)\vec{n} = \frac{2}{6}\vec{n} = \frac{1}{3}\vec{n}$.

Then distance $= \|\vec{p}\| = \|\frac{1}{3}\vec{n}\| = \frac{1}{3}\sqrt{1^2+1^2+(-2)^2} = \frac{\sqrt{6}}{3}$.

4. (3 marks) Let $\mathbf{u} = [-4, 0, 3]$ and $\mathbf{v} = [2, 5, 1]$. Find vectors $\mathbf{p}$ and $\mathbf{q}$ so that $\mathbf{v} = \mathbf{p} + \mathbf{q}$, $\mathbf{p}$ is parallel to $\mathbf{u}$, and $\mathbf{q}$ is orthogonal to $\mathbf{u}$. *[Hint: Use projections to find one of the vectors.]*

$\vec{p} = \text{proj}_{\vec{u}}(\vec{v}) = \left(\frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\right)\vec{u}$

$= \frac{-5}{25}\vec{u} = -\frac{1}{5}[-4,0,3] = [\frac{4}{5}, 0, \frac{-3}{5}]$



$\vec{q} = \vec{v} - \vec{p} = [2,5,1] - [\frac{4}{5}, 0, -\frac{3}{5}] = [\frac{6}{5}, 5, \frac{8}{5}]$.

5. (3 marks) Find an equation, in parametric form, of the line passing through the point $P = (1, 2, -1)$ and orthogonal to the plane defined by

$$\begin{cases} x = 6 + 2s - t \\ y = 1 - 3s + 5t \\ z = -7 + s - t. \end{cases} \Rightarrow \vec{x} = \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + s\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 5 \\ -1 \end{bmatrix}$$

direction vectors for plane

$$\vec{n} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \times \begin{bmatrix} -1 \\ 5 \\ -1 \end{bmatrix} = \left|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 1 \\ -1 & 5 & -1 \end{matrix}\right| \begin{matrix} \hat{i} & \hat{j} \\ 2 & -3 \\ -1 & 5 \end{matrix}$$

$$= 3\hat{i} - \hat{j} + 10\hat{k} - 3\hat{k} - 5\hat{i} + 2\hat{j} = -2\hat{i} + \hat{j} + 7\hat{k}$$

This normal vector is orthogonal to the plane and $\therefore$ a direction vector for the line.

$$\therefore \vec{x} = \vec{p} + t\vec{d} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}$$

In parametric form, $\begin{cases} x = 1 - 2t \\ y = 2 + t \\ z = -1 + 7t \end{cases}$ (Answer not unique.)

6. (3 marks) Find the vector form of the equation of the plane $3x - 4y + 2z = 12$.

Three points on the plane are $A = (4,0,0)$, $B = (0,-3,0)$ and $C = (0,0,6)$.

$\therefore \vec{AB} = \begin{bmatrix} -4 \\ -3 \\ 0 \end{bmatrix}$ and $\vec{AC} = \begin{bmatrix} -4 \\ 0 \\ 6 \end{bmatrix}$ are nonparallel direction vectors for the plane. Scaling to simplify (optional) gives us $\vec{u} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$. Using point A for $\vec{p} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$, we get the vector form $\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$ given by

$$\vec{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$$ (Answer not unique.)

7. (2 marks) Find the value(s) of k such that the vector

$$\mathbf{v} = \begin{bmatrix} -1 \\ k \\ 2 \end{bmatrix}$$

is orthogonal to the plane $2x + 3y - 4z = 0$.

$\vec{v}$ must be parallel to (i.e. a scalar multiple of) the normal vector $\vec{n} = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$ to the plane:

$$\vec{v} = c\,\vec{n} \Rightarrow \begin{bmatrix} -1 \\ k \\ 2 \end{bmatrix} = c\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 2c \\ 3c \\ -4c \end{bmatrix}$$

$\therefore\ -1 = 2c \Rightarrow c = -\tfrac{1}{2}$ and similarly $2 = -4c \Rightarrow c = -\tfrac{1}{2}$

$\therefore\ k = 3c = 3(-\tfrac{1}{2}) = -\tfrac{3}{2}$

OR Find any vector parallel to the plane with a nonzero y-component.

eg Since $A = (0,0,0)$ and $B = (3,-2,0)$ are in the plane then $\vec{AB} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$ is such a parallel vector.

Since $\vec{v}$ must be orthogonal to $\vec{AB}$, then

$$\vec{v} \cdot \vec{AB} = 0 \Rightarrow (-3) - 2k + 0 = 0 \Rightarrow k = -\tfrac{3}{2}.$$