CAMOSUN

Name: SOLUTIONS

Mark: $\overline{}$ **25**

MATH 226 (Winter, 2025)

Test 2

1. (5 marks) Use either the method of undetermined coefficients or annihilators to solve the differential equation $y'' - 6y' + 5y = 8e^x$.

[H] $y'' - 6y' + 5y = 0$

$$m^2 - 6m + 5 = 0$$

$$(m-1)(m-5) = 0$$

$$m = 1, 5$$

$$\therefore y_h = c_1 e^x + c_2 e^{5x}$$

$$y_p = Axe^x$$

$$y_p' = Axe^x + Ae^x$$

$$y_p'' = Axe^x + 2Ae^x$$

$$(\cancel{Axe^x} + 2Ae^x) - 6(\cancel{Axe^x} + Ae^x) + \cancel{5Axe^x} = 8e^x$$

$$-4Ae^x = 8e^x$$

$$A = -2$$

$$\therefore y_p = -2xe^x$$

$$\therefore y = c_1 e^x + c_2 e^{5x} - 2xe^x$$

OR $(D^2 - 6D + 5)y = 8e^x$

$$(D-1)(D-5)y = 8e^x$$

annihilate with $D-1$

$$(D-1)^2(D-5)y = 0$$

$$y = \underbrace{c_1 e^x}_{y_h} + \underbrace{c_2 xe^x}_{y_p} + \underbrace{c_3 e^{5x}}_{y_h}$$

⋮

2. (6 marks) Use variation of parameters to find the general solution of $y'' + y = \sec x \tan x$, starting from

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = f(x) \end{cases}.$$

$$\left.\begin{array}{l} [H]\quad y''+y=0 \\ m^2+1=0 \\ m=\pm i \end{array}\right\} \therefore\ y_h = C_1\cos x + C_2 \sin x$$

$$y_p = u_1 \cos x + u_2 \sin x$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$u_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \sec x \tan x & \cos x \end{vmatrix}}{W} = \frac{-\sin x \sec x \tan x}{1} = -\tan^2 x = 1 - \sec^2 x$$

$$\therefore u_1 = \int (1-\sec^2 x)\,dx = x - \tan x$$

$$u_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \tan x \end{vmatrix}}{W} = \frac{\cos x \sec x \tan x}{1} = \tan x$$

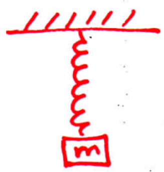
$$\therefore u_2 = \int \tan x\,dx = \ln|\sec x| \quad (\text{or } -\ln|\cos x|)$$

$$\therefore y_p = (x-\tan x)\cos x + (\ln|\sec x|)\sin x$$

$$= x\cos x - \underbrace{\sin x}_{\text{part of } y_h} + (\ln|\sec x|)\sin x$$

$$\therefore y = C_1\cos x + C_2 \sin x + \underbrace{x\cos x}_{\text{or could leave as } x\cos x - \sin x} + (\ln|\sec x|)\sin x$$

3. (4 marks) A 5 kg mass is attached to a spring, causing it to stretch 2 m. The mass is released 1 m below the equilibrium position with an initial upward velocity of 3 m/s and is subject to a damping force that is twice its velocity. Set up, **but do not solve**, the initial-value problem (both differential equation and initial conditions) for the equation of motion $x(t)$ of the mass. Recall $g = 9.8\,\mathrm{m/s^2}$.



$m = 5\,\mathrm{kg}$, $s = 2\,\mathrm{m}$, $\beta = 2\,\mathrm{kg/s}$

$mg = ks \Rightarrow k = \frac{mg}{s} = \frac{5(9.8)}{2} = 24.5\,\frac{N}{m}$

$x'' + \frac{\beta}{m}x' + \frac{k}{m}x = 0$

$\Rightarrow x'' + \frac{2}{5}x' + \frac{24.5}{5}x = 0$

$\Rightarrow \underbrace{x'' + 0.4x' + 4.9x = 0}_{\text{D.E.}}, \quad \underbrace{x(0) = 1\,\mathrm{m}, \quad x'(0) = -3\,\mathrm{m/s}}_{\text{I.C.}}$

4. (5 marks) Find a power series solution of the differential equation $\frac{dy}{dx}+xy=0$ and then convert it to a closed form solution.

$$y=\sum_{n=0}^{\infty} a_n x^n \quad , \quad y'=\sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$\sum_{n=1}^{\infty} a_n n x^{n-1} + x\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} a_{n+1}(n+1)x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$a_1 + \sum_{n=1}^{\infty} a_{n+1}(n+1)x^n + \sum_{n=1}^{\infty} a_{n-1}x^n = 0$$

$$a_1 + \sum_{n=1}^{\infty} \left[a_{n+1}(n+1)+a_{n-1}\right]x^n = 0$$

a_0 free, $a_1=0$, and $a_{n+1}=\frac{-a_{n-1}}{n+1}$ for $n\geq 1$

<u>Evens</u> (odds all 0) (Jump of 2)

$a_2=\frac{-a_0}{2}$

$a_4=\frac{-a_2}{4}=\frac{(-1)^2 a_0}{2\cdot 4}$

$a_6=\frac{-a_4}{6}=\frac{(-1)^3 a_0}{2\cdot 4\cdot 6}$

$a_8=\frac{-a_6}{8}=\frac{(-1)^4 a_0}{2\cdot 4\cdot 6\cdot 8}$

In general,

$$a_{2k}=\frac{(-1)^k a_0}{2\cdot 4\cdot 6\cdots(2k)} = \frac{(-1)^k a_0}{2^k k!} \quad \text{(works for } k=0\text{)}$$

$$\therefore\ y=\sum_{k=0}^{\infty}\frac{(-1)^k a_0}{2^k k!}x^{2k} = a_0\sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{x^2}{2}\right)^k = a_0 e^{-\frac{x^2}{2}}$$

5. (5 marks) Use Laplace transforms to solve the initial-value problem.

$$y'' + 2y' + y = 1 + \delta(t-3),\ y(0) = 0,\ y'(0) = 0$$

$$s^2Y(s) - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} + 2(sY(s) - \cancel{y(0)}^{0}) + Y(s) = \frac{1}{s} + e^{-3s}$$

$$(s^2+2s+1)Y(s) = \frac{1}{s} + e^{-3s}$$

$$(s+1)^2 Y(s) = \frac{1}{s} + e^{-3s}$$

$$Y(s) = \underbrace{\frac{1}{s(s+1)^2}} + \frac{1}{(s+1)^2}e^{-3s}$$

$$\frac{1}{s(s+1)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$$1 = A(s+1)^2 + Bs(s+1) + Cs$$

$s=0 \Rightarrow A = 1$

$s=-1 \Rightarrow C = -1$

$s=1 \Rightarrow 1 = 4A + 2B + C = 4 + 2B - 1$

$\rightarrow 2B = -2 \Rightarrow B = -1$

$$\therefore Y(s) = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2} + \frac{1}{(s+1)^2}e^{-3s}$$

$$\therefore y(t) = 1 - e^{-t} - te^{-t} + (t-3)e^{-(t-3)}u(t-3).$$