Name: SOLUTIONS

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MATH 226 (Winter, 2025)

Test 1

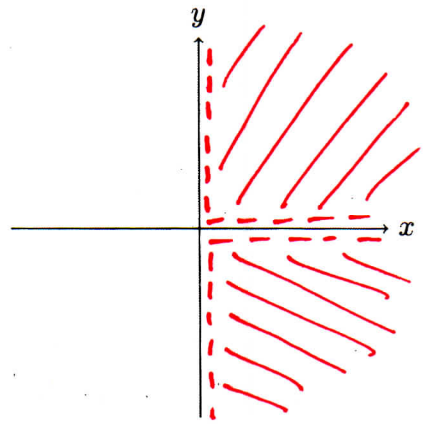
1. Consider the first-order differential equation $\dfrac{dy}{dx} = 9x^{1/2}y^{1/3}$.

   (a) (2 marks) Determine and sketch the region in the xy-plane where the differential equation would have a unique solution through (x_0, y_0).

$f(x,y) = 9x^{1/2}y^{1/3}$ is cont. for $x>0$

$\dfrac{\partial f}{\partial y} = \dfrac{3x^{1/2}}{y^{2/3}}$ is cont for $x>0$ and $y \neq 0$

$\therefore$ Need $x_0 > 0$ and $y_0 \neq 0$ for uniqueness of solutions



   (b) (3 marks) Show that $y = 8x^{9/4}$ is a solution of the differential equation with initial condition $y(0) = 0$. Could there be other solutions? Briefly explain.

$\dfrac{dy}{dx} = 18x^{5/4}$

$9x^{1/2}y^{1/3} = 9x^{1/2}(8x^{9/4})^{1/3} = 9x^{1/2}(2x^{3/4}) = 18x^{5/4}$ $=$

$\therefore$ sol'n of D.E.

Clearly $y = 8x^{9/4}$ satisfies $y(0)=0$ since $8(0)^{9/4} = 0$.

There could be other solutions since $(0,0)$ is not in the region from part (a) where unique solutions are guaranteed.

(Note: $y=0$ is also a solution.)

2. (2 marks) State the order of each differential equation and whether it is linear or nonlinear.

(a) $x\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0$

1st order, nonlinear

(c) $x\left(x+\frac{1}{y}\right)dx + \frac{1}{y}dy = 0$

$\Rightarrow \frac{dy}{dx} + x^2y = -x$

1st order, linear

(b) $(\tan t)\frac{d^3u}{dt^3} + (\cos t)\frac{du}{dt} + te^tu = \sin t$

3rd order, linear

(d) $\frac{d^2y}{dx^2} - y\frac{dy}{dx} + y = 0$

2nd order, nonlinear

3. (3 marks) Solve the initial value problem $\frac{dy}{dx} = \sin x \csc y$, $y(0) = \pi/2$. Express y explicitly as a function of x.

$\int \sin y \, dy = \int \sin x \, dx$

$-\cos y = -\cos x + C$

$y(0) = \frac{\pi}{2} \Rightarrow 0 = -1 + C \Rightarrow C = 1$

$\therefore -\cos y = -\cos x + 1$

$\cos y = \cos x - 1$

$y = \arccos(\cos x - 1)$

4. Solve the following first-order differential equations.

(a) (3 marks) $\dfrac{dy}{dx} = \dfrac{x^2+xy+y^2}{x^2}$ [Express y explicitly as a function of x.]

$\dfrac{dy}{dx} = 1 + \dfrac{y}{x} + \left(\dfrac{y}{x}\right)^2 = 1+u+u^2$ where $u = \dfrac{y}{x}$.

$y = ux \Rightarrow y' = u + u'x$

$\therefore\ u'x + u = 1 + u + u^2$

$u'x = u^2 + 1$

$\displaystyle\int \frac{du}{u^2+1} = \int \frac{1}{x}\,dx$

$\arctan u = \ln|x| + C$

$u = \tan(\ln|x| + c)$

$y = ux \Rightarrow y = x\tan(\ln|x| + c)$

(b) (3 marks) $\underbrace{(2xy + x + ye^{xy})}_{M}dx + \underbrace{(x^2 + y + xe^{xy})}_{N}dy = 0$

$\dfrac{\partial M}{\partial y} = 2x + xye^{xy} + e^{xy}$

$\dfrac{\partial N}{\partial x} = 2x + xye^{xy} + e^{xy}$

$=\ \therefore$ exact

$f = \int M\,dx = \int (2xy + x + ye^{xy})\,dx = x^2y + \frac{1}{2}x^2 + e^{xy} + g(y)$

$f = \int N\,dy = \int (x^2 + y + xe^{xy})\,dy = x^2y + \frac{1}{2}y^2 + e^{xy} + h(x)$

$\therefore\ f(x,y) = x^2y + \frac{1}{2}x^2 + \frac{1}{2}y^2 + e^{xy}$

$\therefore$ soln of D.E. is

$x^2y + \frac{1}{2}x^2 + \frac{1}{2}y^2 + e^{xy} = C$

5. (3 marks) Given that $y_1 = x$ is a solution of $x^2y'' + xy' - y = 0$, use the **reduction of order** method to find a second solution.

Let $y_2 = uy_1 = ux$.

Then $y_2' = u + u'x$

$y_2'' = u' + u' + u''x = 2u' + u''x$

$\therefore x^2(u''x + 2u') + x(u'x + u) - ux = 0$

$x^3u'' + 2x^2u' + x^2u' + \cancel{xu} - \cancel{xu} = 0$

$x^3u'' + 3x^2u' = 0$

$u'' + \frac{3}{x}u' = 0 \qquad \mu(x) = e^{\int \frac{3}{x}dx} = x^3$

$\frac{d}{dx}[x^3u'] = 0 \Rightarrow x^3u' = C_1 \Rightarrow u' = C_1x^{-3}$

$\therefore u = -\frac{C_1}{2}x^{-2} + C_2$

$\Rightarrow y_2 = ux = -\frac{C_1}{2}x^{-1} + C_2x$

or just $y_2 = \frac{1}{x}$ (letting $C_1 = -2, C_2 = 0$)

6. (2 marks) Find the general solution of the differential equation $y^{(4)} + 4y'' = 0$.

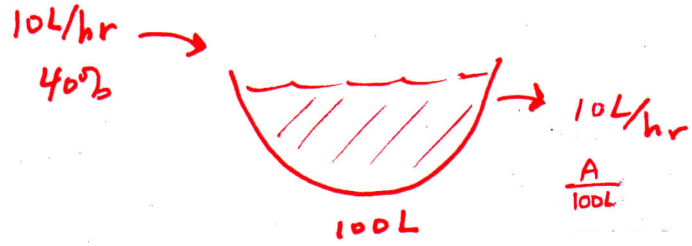
$m^4 + 4m^2 = 0$

$m^2(m^2 + 4) = 0$

$m = 0\ (\times 2), \quad m = \pm 2i$

$\therefore y = C_1 + C_2x + C_3\cos 2x + C_4\sin 2x.$

7. (4 marks) A large 100 liter punch bowl is kept full by adding 10 liters per hour of a 40% alcohol mix. The guests consume 10 liters per hour of the mixed punch. If there were initially 20 liters of alcohol in the bowl, find the amount $A(t)$ of alcohol in the bowl (in liters) at time t (in hours) by setting up and solving a linear differential equation.



$$\frac{dA}{dt} = \text{rate in} - \text{rate out}$$
$$= 40\%\,(10\,L/hr) - \left(\frac{A}{100L}\right)(10\,L/hr)$$
$$= 4 - \frac{1}{10}A$$

$$\frac{dA}{dt} + \frac{1}{10}A = 4 \qquad \mu(t) = e^{\int \frac{1}{10}\,dt} = e^{\frac{1}{10}t}$$

$$\frac{d}{dt}\left[e^{\frac{1}{10}t}A\right] = 4e^{\frac{1}{10}t}$$
$$e^{\frac{1}{10}t}A = 40e^{\frac{1}{10}t} + C$$
$$A = 40 + Ce^{-\frac{1}{10}t}$$
$$A(0) = 20 \Rightarrow 20 = 40 + C \Rightarrow C = -20.$$
$$\therefore\ A(t) = 40 - 20e^{-\frac{1}{10}t}.$$