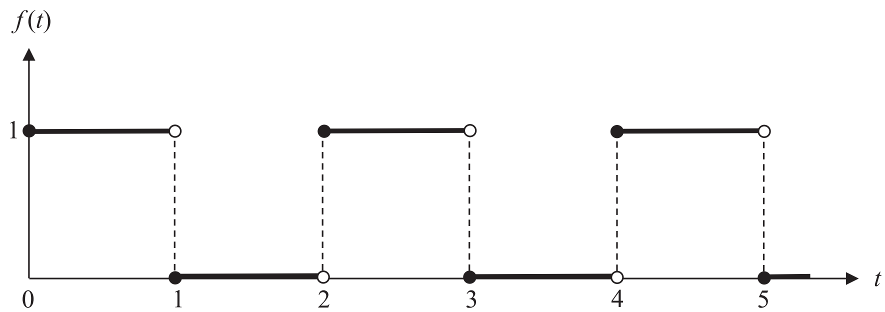
Laplace Transform of a Square Wave

Consider the square wave function

$$f(t) = \begin{cases} 1, & 2k \le t < 2k+1 \text{ for } k = 0,1,2,\dots \\ 0, & 2k+1 \le t < 2k+2 \text{ for } k = 0,1,2,\dots, \end{cases}$$

whose graph is given below. What is its Laplace transform?



Since f is periodic with period $T = 2$, we can use the formula $\mathscr{L}\{f(t)\} = \dfrac{1}{1-e^{-sT}} \displaystyle\int_0^T f(t)e^{-st}\,dt$.

$$\begin{aligned} \mathscr{L}\{f(t)\} &= \frac{1}{1-e^{-2s}} \int_0^2 f(t)e^{-st}\,dt = \frac{1}{1-e^{-2s}} \left[\int_0^1 (1)e^{-st}\,dt + \int_1^2 (0)e^{-st}\,dt \right] \\ &= \frac{1}{1-e^{-2s}} \int_0^1 e^{-st}\,dt = \frac{1}{1-e^{-2s}} \left[-\frac{1}{s}e^{-st} \right]_0^1 = \frac{1}{1-e^{-2s}} \left[-\frac{1}{s}e^{-s} + \frac{1}{s} \right] \\ &= \frac{1-e^{-s}}{s(1-e^{-2s})} = \frac{1-e^{-s}}{s(1+e^{-s})(1-e^{-s})} = \frac{1}{s(1+e^{-s})}. \end{aligned}$$

Using a geometric series we can rewrite this as

$$\mathscr{L}\{f(t)\} = \frac{1}{s}\left[\frac{1}{1-(-e^{-s})}\right] = \frac{1}{s}\sum_{n=0}^{\infty}(-e^{-s})^n = \frac{1}{s}\sum_{n=0}^{\infty}(-1)^n e^{-ns}.$$

Alternatively, if we express $f(t)$ as a series of Heaviside step functions,

$$f(t) = 1 - u(t-1) + u(t-2) - u(t-3) + u(t-4) - \cdots = \sum_{n=0}^{\infty}(-1)^n u(t-n),$$

then we get

$$\mathscr{L}\{f(t)\} = \sum_{n=0}^{\infty}(-1)^n \mathscr{L}\{u(t-n)\} = \sum_{n=0}^{\infty}\frac{(-1)^n e^{-ns}}{s} = \frac{1}{s}\sum_{n=0}^{\infty}(-1)^n e^{-ns}.$$