Power Series

Power Series and Interval of Convergence

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots ,$$

is called a **power series centered at c**.

Every power series has an interval of convergence and corresponding radius of convergence, R, usually found with the help of the Ratio Test, satisfying one of the following three cases.

1. The series converges absolutely for all x. The interval of convergence is $(-\infty, \infty)$ and $R = \infty$.

2. The series converges only at c. In this case the "interval" of convergence degenerates to the singleton point $\{c\}$ and $R = 0$.

3. The series converges absolutely if $|x - c| < R$ and diverges if $|x - c| > R$, where $R > 0$. The interval of convergence may be $(c - R, c + R)$, $[c - R, c + R)$, $(c - R, c + R]$ or $[c - R, c + R]$, depending on the behaviour at the endpoints.

Power Series as a Function

A power series defines a function $f(x)$ according to

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots ,$$

where the domain of f is the interval of convergence of the power series.

If $R > 0$ or $R = \infty$, then f is differentiable and integrable and both $f'(x)$ and $\int f(x)\,dx$ can be found by differentiating or integrating term by term, like a polynomial. Moreover, both $f'(x)$ and $\int f(x)\,dx$ have the same radius R and interval of convergence as $f(x)$, except that in the case where $0 < R < \infty$, the behaviour at the endpoints $x = c \pm R$ may differ.

Differentiating gives

$$f'(x) = \sum_{n=1}^{\infty} a_n n(x-c)^{n-1} = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots ,$$

Note that indexing here starts at $n = 1$ since the derivative of the constant term a_0 is zero and so the first term corresponding to $n = 0$ disappears. Re-indexing the series by replacing n by $n + 1$ allows the series to be written

$$f'(x) = \sum_{n=0}^{\infty} a_{n+1}(n+1)(x-c)^n.$$

The second derivative can be similarly found

$$f''(x) = \sum_{n=2}^{\infty} a_n n(n-1)(x-c)^{n-2} = 2a_2 + 6a_3(x-c) + 12a_4(x-c)^2 + \cdots .$$

This time indexing starts at $n = 2$ since the second derivatives of the first two terms a_0 and $a_1(x-c)$ of $f(x)$ are both zero. Re-indexing the series by replacing n by $n + 2$ allows the series to be written

$$f''(x) = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x-c)^n.$$

Integrating $f(x)$ gives

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1} = C + a_0(x-c) + \frac{a_1}{2}(x-c)^2 + \frac{a_2}{3}(x-c)^3 + \cdots .$$

Analytic Functions and Taylor Series

If a function $f(x)$ can be represented by a power series centered at c with $R > 0$ or $R = \infty$, then f is said to be **analytic at c**. In this case, f will equal its Taylor series (or Maclaurin series if $c = 0$) on its interval of convergence,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots .$$

The following is a list of some special Taylor Series.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n \qquad -1 < x < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \qquad -\infty < x < \infty$$

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{(n+1)}(x-1)^n}{n} \qquad 0 < x \leq 2$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad -\infty < x < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \qquad -\infty < x < \infty$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} x^{2n+1} \qquad -1 \leq x \leq 1$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1} \qquad -\infty < x < \infty$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n} \qquad -\infty < x < \infty$$

Arithmetic of Power Series

Power series can be scalar multiplied, added, subtracted or multiplied as if they were polynomials. For example, in the case of power series centered at $c = 0$,

$$k \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} k a_n x^n \qquad \text{and} \qquad \sum_{n=0}^{\infty} a_n x^n \pm \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n,$$

where k is any real number. Multiplying power series gives

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots)$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + (a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0)x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} x^n$$