C A M O S U N C O L L E G E

MATHEMATICS 226

NON-LINEAR DIFFERENTIAL EQUATIONS AND THE PHASE PLANE

TOPICS:

I. Plane Autonomous Systems.

II. The Non-linear Pendulum.

I. PLANE AUTONOMOUS SYSTEMS

INTRODUCTION

Consider the motion of a single particle in a two dimensional fluid. Let $\mathbf{v} = P(x, y, t)\mathbf{i} + Q(x, y, t)\mathbf{j}$ be the velocity vector field of the fluid which we will assume is known. That is, we know the velocity at every point (x, y) in the fluid at any time t. Our problem is to find the motion of a particle as it is carried along with the fluid. If we start the particle at $(x_{0,} y_0)$ and let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ be the position of the particle , then we must find $x(t)$ and $y(t)$ so that $\frac{d\mathbf{r}}{dt} = \mathbf{v}$. Such a trajectory is called an integral curve of the vector field $\mathbf{v}$. One should also note that the field vectors along the curve are tangent. To find such a curve we are lead to solve the first order system:

$$\begin{aligned} \frac{dx}{dt} &= P(x, y, t) \quad x(0) = x_0 \\ \frac{dy}{dt} &= Q(x, y, t) \quad y(0) = y_0 \end{aligned} . \qquad [1]$$

These equations may be non-linear and very difficult to solve. The problem might be easier to solve if P and Q did not depend on t (steady fluid flow). That is, the equations do not explicitly contain t. When this is the case, the system is called an autonomous system. If the system is autonomous, then we can eliminate t completely by dividing the second differential equation by the first. The problem then reduces to solving a single first order equation for $y(x)$:

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)} \text{ with } y(x_0) = y_0 .$$

The solution $y(x)$ is not, however, the full solution of the system of equations we started with. We lost information about the motion when we eliminated t . $y(x)$ gives only the path or "orbit" of the particle. The original solution, $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ would have given the orbit as well as the location of the particle on the orbit at any time t. In cases where the autonomous system is too difficult to solve, we can still learn much from the orbits.

We will be interested in autonomous systems of the form:

$$\frac{dx}{dt} = P(x, y)$$
$$\frac{dy}{dt} = Q(x, y)$$

called plane autonomous systems.

EXAMPLE 1:

Find the orbits of the system

$$\frac{dx}{dt} = x$$
$$\frac{dy}{dt} = 2y$$

Solution:

$$\frac{dy}{dx} = \frac{2y}{x} \text{ or } xdy - 2ydx = 0 \text{ or } \frac{x^2dy - 2xydx}{x^4} = d\left(\frac{y}{x^2}\right) = 0 \text{ so that } y = cx^2 \text{ .}$$

The orbits are the family of parabolas $y = cx^2$ including $y \equiv 0$ and don't forget the other solution $x \equiv 0$ (where from?).

This system is simple enough to have directly solved for $x(t)$ and $y(t)$ obtaining

$$x(t) = c_1e^t \text{ and } y(t) = c_2e^{2t}.$$

CRITICAL POINTS

A critical point (or equilibrium point) of the plane autonomous system [1] is a point where $P(x, y) = 0$ and $Q(x, y) = 0$. Critical points are points where the particle in the fluid example would be stopped. Notice that $\frac{dy}{dx}$ is undefined (direction of orbit undefined). Critical points are of physical interest in many problems because they represent steady state or equilibrium solutions.

Suppose (x_1, y_1) is a critical point of [1]. It is easy to see that the constant functions $x(t) \equiv x_1$, $y(t) \equiv y_1$ are solutions of the system. Since the velocity is zero at this point, the particle will stay at the point forever. The important question about critical points is not what happens at them but what happens nearby. What happens to the particle if it is near the critical point? Does it get carried to or away from the point? In example 1 above, (0,0) is a critical point. Since e^t and e^{2t} increase with t, the particle gets carried away from the critical point. We would call this and unstable equilibrium point.

STABILITY

A critical point is a stable critical point if whenever the particle is on a trajectory near the critical point, it stays near the critical point for all time. The particle may go to the critical point and stay there, it may asymptotically spiral into the critical point, or it may keep its distance and orbit the critical point. All of these types of behavior are termed stable. If you can find **any** trajectory near the critical point which will take the particle forever away from the critical point, then the point is unstable.

EXAMPLE 2:

Find the orbits of the system and determine the nature of the critical point.

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x$$

Solution:

$$\frac{dy}{dx} = -\frac{x}{y} \quad \text{or} \quad xdx + ydy = 0 \quad \text{so} \quad x^2 + y^2 = c$$

and the orbits are circles with centres at the critical point (0,0). The critical point is stable.

EXAMPLE 3:

A mass m on a spring with spring constant k satisfies the second order differential equation:

$$m\frac{d^2x}{dt^2} = -kx.$$

If we let $p = m\frac{dx}{dt}$, then we have the first order system:

$$\frac{dx}{dt} = \frac{p}{m}$$

$$\frac{dp}{dt} = -kx$$

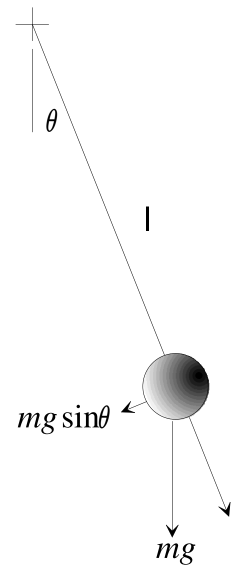
where the new variable p is the momentum of the mass. This is an autonomous system with $(x, p) = (0,0)$ as a critical point. Dividing the second equation by the first, we find:

$$\frac{dp}{dx} = -\frac{kmx}{p} \quad \text{or} \quad \frac{p}{m}dp + kxdx = 0 \quad \text{or} \quad \frac{p^2}{2m} + \frac{1}{2}kx^2 = C \quad ,$$

which says two important things about the system. First, the orbits are ellipses and second, the total energy is constant (K.E. + P.E. = C).

The plane of the problem does not correspond to the physical space in which the mass moves. Instead, one dimension is the momentum p of the mass and the other is the position x. This is called the phase plane of the system. The mass-spring system is represented by a single point in its phase plane. The point orbits around the ellipse as the mass bounces back and forth on the spring. The state of the system is completely determined and represented by the position of its point in the phase plane.

II. THE NON-LINEAR PENDULUM



Consider a pendulum of length ℓ and mass m. If θ measures the angle from the vertical, the differential equation is:

$$m\ell^2 \frac{d^2\theta}{dt^2} = -mg\ell \sin\theta.$$

Let $L = m\ell^2 \frac{d\theta}{dt}$ so that we have the following system:

$$\frac{d\theta}{dt} = \frac{L}{m\ell^2}$$

$$\frac{dL}{dt} = -mg\ell \sin\theta$$

The system has infinitly many critical points at $(\theta, L) = (n\pi, 0)$. To find the orbits we eliminate t to find:

$$\frac{dL}{d\theta} = -\frac{m^2 g\ell^3 \sin\theta}{L} \quad \text{or} \quad \frac{L}{m\ell^2} dL + mg\ell \sin\theta d\theta = 0 \quad \text{or} \quad \frac{L^2}{2m\ell^2} - mg\ell \cos\theta = C.$$

The orbits are plotted here for different values of C. (Energy).