Partial Derivatives

The derivative of a function $y = f(x)$ of one independent variable x is defined by

$$\frac{d}{dx}f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x},$$

if the limit exists. The derivative can be also be denoted by $f'(x)$, y', $\dfrac{dy}{dx}$, or $D_x f(x)$.

For a function $z = f(x, y)$ of two independent variables x and y, its *partial derivatives* (or simply *partials*) with respect to each variable are given by

$$\frac{\partial}{\partial x}f(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x},$$

$$\frac{\partial}{\partial y}f(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y},$$

if these limits exist. The partial derivative of f with respect to x is found by holding y constant and computing an ordinary derivative with respect to x. Likewise, the partial derivative of f with respect to y is found by holding x constant and computing an ordinary derivative with respect to y. Other notation for partial derivatives include

$$\frac{\partial}{\partial x}f(x,y) = \frac{\partial f}{\partial x} = f_x(x,y) = f_x = z_x = \frac{\partial z}{\partial x} = D_x f = D_x f(x,y),$$

$$\frac{\partial}{\partial y}f(x,y) = \frac{\partial f}{\partial y} = f_y(x,y) = f_y = z_y = \frac{\partial z}{\partial y} = D_y f = D_y f(x,y).$$

Example 1: The partial derivatives of $f(x, y) = x^4 + 2x^2y^3 - y^2 + 7$ are $f_x = 4x^3 + 4xy^3$ and $f_y = 6x^2y^2 - 2y$.

Partial derivatives can similarly be defined for functions of three or more variables by differentiating with respect to one variable and holding the others constant.

Example 2: The partial derivatives of $f(x, y, z) = 5xy^2z^3$ are $f_x = 5y^2z^3$, $f_y = 10xyz^3$ and $f_z = 15xy^2z^2$.

Just as one can compute second, third, and higher order derivatives of $f(x)$, one can also compute higher order partial derivatives. A function $f(x, y)$ will have four second partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = (f_x)_x = f_{xx} \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = (f_y)_y = f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = (f_y)_x = f_{yx} \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = (f_x)_y = f_{xy}$$

Example 3: The second partial derivatives of the function $f(x, y) = x^4 + 2x^2y^3 - y^2 + 7$ from Example 1 are $f_{xx} = 12x^2 + 4y^3$, $f_{yy} = 12x^2y - 2$, $f_{yx} = 12xy^2$, and $f_{xy} = 12xy^2$.

Note that the mixed partials f_{xy} and f_{yx} are equal in the previous example. This property is true in general when they are continuous.