Ordinary and Singular Points

A homogeneous second-order linear differential equation has the form

$$A_2(x)y'' + A_1(x)y' + A_0(x)y = 0. \tag{1}$$

By dividing through by $A_2(x)$, we obtain the standard form

$$y'' + P(x)y' + Q(x)y = 0. \tag{2}$$

A point $x = x_0$ is said to be an **ordinary point** of the differential equation (1) if both $P(x)$ and $Q(x)$ in (2) are analytic at x_0. Otherwise, it is called a **singular point**.

Theorem 6.2.1 (Existence of Power Series Solutions)
If $x = x_0$ is an ordinary point of the differential equation (1), we can always find two linearly independent power series solutions centered at x_0 of the form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

A power series solution converges at least on some interval defined by $|x - x_0| < R$, where R is the distance from x_0 to the nearest singular point. □

A singular point is called a **regular singular point** if the functions $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2 Q(x)$ are both analytic at x_0. Otherwise, it is called an **irregular singular point**. For regular singular points, $p(x)$ and $q(x)$ have Taylor series centered at x_0,

$$p(x) = \sum_{n=0}^{\infty} p_n (x - x_0)^n \quad \text{and} \quad q(x) = \sum_{n=0}^{\infty} q_n (x - x_0)^n,$$

and so

$$\begin{aligned} P(x) &= \frac{p_0}{x - x_0} + p_1 + p_2(x - x_0) + p_3(x - x_0)^2 + \cdots, \quad \text{and} \\ Q(x) &= \frac{q_0}{(x - x_0)^2} + \frac{q_1}{x - x_0} + q_2 + q_3(x - x_0) + q_4(x - x_0)^2 + \cdots. \end{aligned}$$

Theorem 6.3.1 (Frobenius' Theorem)
If $x = x_0$ is a regular singular point of the differential equation (1), then there exists at least one solution of the form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r},$$

where r is a constant to be determined. The series will converge at least on some interval $0 < x - x_0 < R$. □